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On nonlocal parabolic steady-state equations of cooperative or competing systems

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ABSTRACT

Some systems of parabolic equations with nonlocal initial conditions are studied. The systems arise when considering steady-state solutions to diffusive age-structured cooperative or competing species. Local and global bifurcation techniques are employed to derive a detailed description of the structure of positive coexistence solutions.

1. Introduction and main results

In this paper, we characterize the structure of positive solutions to certain systems of coupled parabolic equations with nonlocal initial conditions. Such systems arise as steady-state equations of two age-structured diffusive populations which inhabit the same spatial region. The interaction between the two species is of cooperative, competing, or predator–prey type leading to different structures of positive solutions. Let the densities of the two species be denoted by $u = u(a, x) \geq 0$ and $v = v(a, x) \geq 0$ with $a \in (0, a_m)$ and $x \in \Omega \subset \mathbb{R}^n$ referring to age and spatial positions, respectively. The models we shall focus on are of the form

$$\partial_a u - \Delta_D u = -\alpha_1 u^2 \pm \alpha_2 vu, \quad a \in (0, a_m),\ x \in \Omega, \tag{1.1}$$

$$\partial_a v - \Delta_D v = -\beta_1 v^2 \pm \beta_2 uv, \quad a \in (0, a_m),\ x \in \Omega, \tag{1.2}$$

subject to the nonlocal initial conditions

$$u(0, x) = \eta \int_0^{a_m} b_1(a)\, u(a, x)\mathrm{d}a, \quad x \in \Omega, \tag{1.3}$$

$$v(0, x) = \xi \int_0^{a_m} b_2(a)\, v(a, x)\mathrm{d}a, \quad x \in \Omega. \tag{1.4}$$

The equations are the steady-state equations of the corresponding time-dependent age-structured equations with diffusion. We refer to [1] for a recent survey on the formidable literature about age-structured population models.

The operator $-\Delta_D$ in (1.1), (1.2) stands for the negative Laplacian on Ω with subscript D indicating that Dirichlet conditions

$$u(a, x) = v(a, x) = 0, \quad a \in (0, a_m),\ x \in \partial\Omega,$$

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are imposed on the smooth boundary $\partial\Omega$ of the bounded domain Ω. The diffusion coefficients in (1.1), (1.2) are taken to be 1, which is a purely notational simplification and does not affect the subsequent mathematical analysis. The number $a_m > 0$ denotes the maximal age of the species. Eqs. (1.3), (1.4) represent the age-boundary conditions reflecting that individuals with age zero are those created when a mother individual of any age $a \in (0, a_m)$ gives birth with rates $\eta b_1(a)$ or $\xi b_2(a)$. The functions $b_j = b_j(a) \geq 0$ describe the profiles of the fertility rates while the parameters $\eta, \xi > 0$ measure their intensity without affecting the structure of the birth profiles. For easier statements of the results we assume the birth profiles

$$b_j \in L_\infty^+((0, a_m)) \quad \text{with } b_j(a) > 0 \text{ for } a \text{ near } a_m,\ j = 1, 2, \tag{1.5}$$

are normalized such that

$$\int_0^{a_m} b_j(a) \mathrm{e}^{-\lambda_1 a}\, \mathrm{d}a = 1, \quad j = 1, 2, \tag{1.6}$$

where $\lambda_1 > 0$ denotes the principal eigenvalue of $-\Delta_D$ on Ω.

Assuming $\alpha_1, \alpha_2, \beta_1, \beta_2 > 0$, the form of the interaction between the two species is determined by the signs on the right hand side of Eqs. (1.1), (1.2). Replacing $\pm$ by a positive sign $+$ in both of the Eqs. (1.1) and (1.2) corresponds to a system (see (1.11), (1.12)) where the two species are *cooperative*, while the case with $\pm$ replaced by negative signs $-$ in each Eq. (1.1) and (1.2) (see (1.14), (1.15)) reflects a *competition* of the species. The case with mixed signs, e.g. a negative sign $-$ in (1.1) instead of $\pm$ and a positive sign $+$ in (1.2) describes a *predator–prey*-system (see (1.17), (1.18)) with a prey density u and a predator density v.

The understanding of the qualitative dynamics of populations requires precise information about equilibrium solutions, i.e. solutions to (1.1)–(1.4). Since obviously $(u, v) \equiv (0, 0)$ solves these equations, the main goal is to establish nontrivial nonnegative steady states and in particular positive *coexistence solutions*, that is, of solutions (u, v) with both components u and v positive and nontrivial. Clearly, the main obstacle are the nonlocal boundary conditions (1.3), (1.4), which, for instance, rule out a straightforward application of the parabolic maximum principle. Suitable maximum principles, however, are given in Lemmas A.1 and A.2 of the Appendix.

This case of a predator–prey system was studied in [2] and local and global bifurcation phenomena of positive nontrivial solutions with respect to the parameters η and ξ were obtained. In the present paper, we shall derive global bifurcation results for the cooperative and the competition case. Depending on η and ξ we shall give a rather complete description of the coexistence solutions. Moreover, we shall also improve the local bifurcation result [2, Theorem 2.4] to a global one.

We like to point out that variants of the elliptic counterparts to (1.1)–(1.2) when age structure is neglected from the outset have been intensively studied in the past, e.g. see [3–17]. Concerning equations for a single specie, e.g. variants of (1.1) subject to (1.3), we refer to [18–23].

To state our results for the present situation, we shall keep ξ fixed and regard η as bifurcation parameter in the following. We thus write (η, u, v) for solutions to (1.1)–(1.4) with u, v belonging to the positive cone $\mathbb{W}_q^+$ of

$$\mathbb{W}_q := L_q((0, a_m), W_{q,D}^2(\Omega)) \cap W_q^1((0, a_m), L_q(\Omega))$$

for $q > n + 2$ fixed, but remark that all our solutions will have smooth components u, v with respect to both $a \in J$ and $x \in \Omega$. We say that a continuum $\mathfrak{C}$ (i.e. a closed and connected set) in $\mathbb{R}^+ \times \mathbb{W}_q^+ \times \mathbb{W}_q^+$ of solutions (η, u, v) to (1.1)–(1.4) is *unbounded with respect to* η, provided the η-projection of $\mathfrak{C}$ contains an interval of the form (η_0, ∞) with $\eta_0 \in \mathbb{R}^+$, and we say that $\mathfrak{C}$ is *unbounded with respect to the u-component in* $\mathbb{W}_q$ provided there is a sequence $((\eta_j, u_j, v_j))_{j\in\mathbb{N}}$ in $\mathfrak{C}$ with $\|u_j\|_{\mathbb{W}_q} \to \infty$ as $j \to \infty$. An analogous terminology shall be used if $\mathfrak{C}$ is unbounded with respect to the v-component.

Clearly, problem (1.1)–(1.4) decouples when taking either u or v to be zero. Noticing that Theorem A.4 from the Appendix provides for each $\eta > 1$ a unique solution $u_\eta \in \mathbb{W}_q^+ \setminus \{0\}$ to

$$\partial_a u - \Delta_D u = -\alpha_1 u^2, \qquad u(0, \cdot) = \eta \int_0^{a_m} b_1(a) u(a, \cdot) \mathrm{d}a, \tag{1.7}$$

and similarly for each $\xi > 1$ a unique solution $v_\xi \in \mathbb{W}_q^+ \setminus \{0\}$ to

$$\partial_a v - \Delta_D v = -\beta_1 v^2, \qquad v(0, \cdot) = \xi \int_0^{a_m} b_2(a)\, v(a, \cdot) \mathrm{d}a, \tag{1.8}$$

there is, independent of what the signs $\pm$ in (1.1), (1.2) are, for any $\xi \geq 0$ the trivial branch

$$\mathfrak{B}_0 := \{(\eta, 0, 0);\ \eta \geq 0\}$$

and the semi-trivial branch

$$\mathfrak{B}_1 := \{(\eta, u_\eta, 0);\ \eta > 1\} \subset \mathbb{R}^+ \times (\mathbb{W}_q^+ \setminus \{0\}) \times \mathbb{W}_q^+ \tag{1.9}$$

of solutions. For $\xi > 1$, an additional semi-trivial branch

$$\mathfrak{B}_2 := \{(\eta, 0, v_\xi);\ \eta \geq 0\} \subset \mathbb{R}^+ \times \mathbb{W}_q^+ \times (\mathbb{W}_q^+ \setminus \{0\}) \tag{1.10}$$

exists. Depending on the signs $\pm$ in (1.1), (1.2) we shall establish further local and global bifurcation of coexistence solutions from these semi-trivial branches.