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Monotone iterative sequences for nonlinear integro-differential equations of second order

Qasem M. Al-Mdallal

Department of Mathematical Sciences, United Arab Emirates University, P. O. Box 17551, Al-Ain, United Arab Emirates

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ABSTRACT

This paper presents an efficient algorithm based on a monotone method for the solution of a class of nonlinear integro-differential equations of second order. This method is applied to derive two monotone sequences of upper and lower solutions which are uniformly convergent. Theorems which list the conditions for the existence of such sequences are presented. The numerical results demonstrate reliability and efficiency of the proposed algorithm.

1. Introduction

In this paper, we consider a class of boundary value problems for second-order nonlinear integro-differential equations of the form

$$\mathcal{L}y := y''(x) + \int_0^x K(x,t)f(y)\,\mathrm{d}t + h(x) = 0, \quad x \in I = [0,1], \tag{1.1}$$

subject to

$$y(0) = y_0, \qquad y(1) = y_1, \tag{1.2}$$

where $f \in C[\mathbb{R}, \mathbb{R}]$ is a decreasing function, $K \in C[I \times I, \mathbb{R}^+]$ is a positive kernel, $h(x) \in C[I, \mathbb{R}]$ and $y_0, y_1 \in \mathbb{R}$. For details about the existence and uniqueness of solutions for such problems, see [1,2].

There has been continuing interest in the solution of nonlinear integro-differential equations due to its importance in studying many models in various scientific fields such as fluid dynamics, plasma physics, mathematical biology and chemical kinetics (for details, see [3] and the references therein).

In practice, analytical explicit solutions for most nonlinear integro-differential equations are usually unavailable and not easy to obtain. Therefore, several numerical techniques have been proposed in the literature to find approximated solutions for nonlinear integro-differential equations. For instance, the Adomian decomposition method [4,3], Homotopy perturbation method [5,6], the spline collocation method [7,8,5], Haar wavelets [9], Tau method [10], Taylor polynomials [11,12], the combined Laplace transform–Adomian decomposition method [13] and the method of upper and lower solutions [14], and the references therein.

The purpose of this paper is to employ an efficient method based on the lower and upper solutions (see [15]), to construct two sequences of decreasing upper solutions, $\{S_k\}$, and increasing lower solutions, $\{s_k\}$, which are uniformly convergent to the solution of Eqs. (1.1)–(1.2). Theoretical analysis of the existence and convergence of those sequences are discussed. The simplicity, reliability and efficiency of the proposed scheme are demonstrated by discussing two numerical examples. It should be noted that the present work is partially an extension to the approach described in [16] in order to solve a class of elliptic equations.

E-mail address: q.almdallal@uaeu.ac.ae.

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The rest of the paper is organized as follows. Some preliminary results are presented in Section 2. In Section 3, numerical examples are discussed to prove the efficiency and the rapid convergence of the present algorithm.

2. Preliminary results

The definitions of lower and upper solutions of the problem (1.1)–(1.2) are given by

Definition 2.1. A function $w \in C^2[I, \mathbb{R}]$ is called a lower solution of (1.1)–(1.2) on I if

$$\mathcal{L}w := w'' + \int_0^x K(x,t) f(w)\, \mathrm{d}t + h(x) \geq 0, \quad x \in I$$

with

$$w(0) \leq y_0, \qquad w(1) \leq y_1,$$

and an upper solution, if the reversed inequalities hold.

Definition 2.2. If $w, v \in C^2[I, \mathbb{R}]$ are, respectively, lower and upper solutions of (1.1)–(1.2) on I with $w(x) \leq v(x)$ for all $x \in I$, then we say that w and v are ordered lower and upper solutions.

In the present study, we assume that an initial ordered lower and upper solutions w and v of (1.1)–(1.2) on I with $w(x) \leq v(x)$ for all $x \in I$ are known. The initials w and v can be constructed by several techniques such as polynomial bounds, eigenfunction expansion bounds or by linearizing the nonlinear part in the problem, for more details see [17,15].

Now, we present some theoretical results that should be utilized to construct two monotone sequences of lower and upper solutions of problem (1.1)–(1.2) on I. In what follows, $[w, v] = \left\{u \in C^2(I) : w \leq u \leq v\right\}$.

Lemma 2.1. *Consider the nonlinear integro-differential equation (1.1)–(1.2) with $f(y)$ is decreasing and $K \geq 0$ in D. If w and v are solutions to (1.1)–(1.2) with $w(x) \leq v(x)$ for all $x \in I$ then $w = v$ on I.*

Proof. We shall prove that $w(x) \geq v(x)$ for all $x \in I$. Since w and v are solutions to (1.1)–(1.2), we have

$$w''(x) + \int_0^x K(x,t) f(w)\, \mathrm{d}t + h(x) = 0, \quad x \in I = [0, 1], \tag{2.1}$$

$$w(0) = y_0, \qquad w(1) = y_1, \tag{2.2}$$

$$v''(x) + \int_0^x K(x,t) f(v)\, \mathrm{d}t + h(x) = 0, \quad x \in I = [0, 1], \tag{2.3}$$

$$v(0) = y_0, \qquad v(1) = y_1. \tag{2.4}$$

Subtracting Eq. (2.1) from Eq. (2.3), we obtain

$$(v - w)''(x) = \int_0^x K(x,t) \left(f(w) - f(v)\right) \mathrm{d}t. \tag{2.5}$$

Since $w \leq v$ and f is decreasing, the integrand $K(x,t)(f(w) - f(v))$ must be positive on the interval $[0, x]$. If we set $Z = v - w$ then $Z'' \geq 0$ with $Z(0) = Z(1) = 0$. It follows from the Maximum Principle that $Z \leq 0$ and, therefore, $v \leq w$ in I as desired. □

Lemma 2.2. *Consider the nonlinear integro-differential equation (1.1)–(1.2) with $f(y)$ is decreasing and $K \geq 0$ in D. Let $g_1(u)$ and $G_1(u)$ be two decreasing functions in the strip $[w, v]$ with $g_1(u) \leq f(u) \leq G_1(u)$. Let s_1 and S_2 be satisfying*

$$\mathcal{L}s_1 := s_1'' + \int_0^x K(x,t) g_1(v)\, \mathrm{d}t + h(x) = 0, \quad x \in I \tag{2.6}$$

$$s_1(0) \leq y_0, \qquad s_1(1) \leq y_1, \tag{2.7}$$

$$\mathcal{L}S_1 := S_1'' + \int_0^x K(x,t) G_1(w)\, \mathrm{d}t + h(x) = 0, \quad x \in I \tag{2.8}$$

$$S_1(0) \geq y_0, \qquad S_1(1) \geq y_1. \tag{2.9}$$

If $w \leq s_1$ and $S_1 \leq v$ in I, then

(i) $s_1, S_1 \in [w, v]$.
(ii) *s_1 and S_1 are, respectively, ordered lower and upper solutions of (1.1)–(1.2) on I.*