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Life span and a new critical exponent for a quasilinear degenerate parabolic equation with slow decay initial values

Chunlai Mu [a,*], Yuhuan Li [b], Ying Wang [c]

[a] College of Mathematics and Physics, Chongqing University, Chongqing, 400044, PR China

[b] Department of Mathematics, Sichuan Normal University, Chengdu 610066, PR China

[c] School of Applied Mathematics, University of Electronic Science and Technology of China, Chengdu 610054, PR China

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ABSTRACT

In this paper, we consider the positive solution of a Cauchy problem for the following *P*-Laplace parabolic equation

$$u_t = \operatorname{div}(|\nabla u|^{p-2}\nabla u) + u^q, \quad p > 2,\ q > 1,$$

and give a secondary critical exponent on the decay asymptotic behavior of an initial value at infinity for the existence of global and nonglobal solutions of the Cauchy problem. Furthermore, the life span of solutions is also studied.

1. Introduction

In 1966, Fujita (see [6]) considered the following initial value problem

$$\begin{aligned} &u_t = \Delta u + u^p, \qquad x \in R^N, \quad t > 0, \\ &u(x,0) = u_0(x), \qquad x \in R^N, \end{aligned} \tag{1.1}$$

where $N \geq 1, p > 1$ and $u_0(x)$ is a bounded positive continuous function. He has shown that there is a critical exponent $p_1^* = 1 + \frac{2}{N}$ such that the solution $u(x,t)$ of (1.1) blows up in finite time for all $u_0(x)$ if $1 < p < p_1^*$; and there are global solutions and nonglobal solutions if $p > p_1^*$. The number p_1^* is the so-called Fujita critical exponent. In [15,28], Hayakawa and Weissler have also proved that p_1^* belongs to the blow-up case. In fact, a similar Fujita's critical exponent for the following Cauchy problem is also given as $p_m^* = m + \frac{2}{N}$.

$$\begin{aligned} &u_t = \Delta u^m + u^p, \quad x \in R^N,\ t > 0 \\ &u(x,0) = u_0(x), \quad x \in R^N, \end{aligned} \tag{1.2}$$

where $p > 1, m > 1$ or $1 > m > \max\{0, 1 - \frac{2}{N}\}$ and $u_0(x)$ is nonnegative bounded and continuous function. In [7,10,24] they have proved that the solution $u(x,t)$ of (1.2) blow up if $1 < p < p_m^*$; while both global and non-global positive solutions exist if $p > p_m^*$. For the case of the critical exponent $p = p^* = m + \frac{2}{N}$, in [21,22] Mochizuki, Mukai and Suzuki have proved that the solutions of (1.2) blow up in finite time (see also [9,11,27]).

* Corresponding author.
E-mail address: chunlaimu@yahoo.com.cn (C. Mu).

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In particular, when $m > 1$, after the transformation $u(x, t) = av^m(bx, t)$, $a = m^{m/(p-1)}$, $b = m^{(p-m)/2(p-1)}$, the equation in (1.2) is translated to

$$v_t = v^\alpha \Delta v + v^\beta, \tag{1.3}$$

where $\alpha = \frac{m-1}{m} \in (0, 1)$ and $\beta = \frac{m+p-1}{m} \in (1, +\infty)$, and one of the results derived in [10] reads as follows

(1) For $1 \le \beta < \alpha + 1 + 2/N(1 - \alpha)$ there are no global positive solutions.
(2) For $\beta > \alpha + 1 + 2/N(1 - \alpha)$, there are both global solution and solutions blowing up in finite time.

Recently, in [31–33] Winkler dropped the restriction $\alpha \in (0, 1)$ in (1.3) by considering general positive p, and studied the following Cauchy problem

$$\begin{aligned} &u_t = u^p \Delta u + u^q, \quad x \in R^N,\ t > 0 \\ &u(x, 0) = u_0(x), \quad x \in R^N \end{aligned} \tag{1.4}$$

where $u_0 \in C^0(R^N) \cap L^\infty(R^N)$ positive and $p \ge 1$ as well as $q \ge 1$. That the exponent $p = 1$ indeed appears as some kind of turning point for the diffusion coefficient in degenerate parabolic equations not in divergence form is already indicated in [13,20,31] where it is proved that the support of a weak solution to (1.4) does not increase with time if $p \ge 1$. This behavior, contrary to the case $p < 1$, may be interpreted as a consequence of the fact that near points where u in small, diffusion is weakened more effectively when $p \ge 1$. Moreover, in [33] he obtained the following results:

(i) For $1 \le q < p + 1$ (resp. $1 \le q < \frac{3}{2}$ if $p = 1$) all positive solutions of (1.4) are global but unbounded provided that $u_0(x)$ decreases sufficiently fast in space.
(ii) For $q = p + 1$, all positive solutions of (1.4) blow up in finite time.
(iii) For $q > p + 1$, there are both global and non-global positive solutions, depending on the size of $u_0(x)$.

It follows from (i)–(iii) that there is a critical growth exponent $q_c = p + 1$ for (1.4) which now, however, has a slightly different meaning and is independent of the space dimension N. Moreover, unlike Eqs. (1.1) and (1.2), the Eq. (1.4) for $p \ge 1$ has the property that the critical exponent q_c would be the same if we replaced R^N with any smooth bounded domain $\Omega \subset R^N$; namely, for this case, the results in [29,31] imply global existence for $1 \le q < p + 1$ and the proof of Theorem 5.1 in [33] have shown that both global existence and finite time blow-up may occur in Ω if $q > p + 1$. The critical exponent $q = p + 1$ in bounded domain is more subtle (see [5,11,30]).

In [8], Galaktionov studied the Cauchy problem of the following quasilinear degenerate parabolic equation

$$\begin{aligned} &u_t = \operatorname{div}(|\nabla u|^{p-2}\nabla u) + u^q, \quad (x, t) \in R^N \times (0, T), \\ &u(x, 0) = u_0(x), \quad x \in R^N, \end{aligned} \tag{1.5}$$

where $p > 2$, $q > 1$ and $u_0(x)$ is a nonnegative bounded and continuous function. Eq. (1.5) is the prototype for a certain class of degenerate equations and appears in the theory of non-Newtonian fluids (see [1]). It is known that the equation is degenerate in the sense that it is not parabolic when $\nabla u = 0$ and shares the same property of heat localization (see [8]). In particular, there is a similar Fujita type result proved in [8] which reads as follows:

(α) If $1 < q < p - 1 + \frac{p}{N}$, then the solution $u(x, t)$ of (1.5) blows up in finite time if $u_0(x) \ne 0$;

(β) If $q > p - 1 + \frac{p}{N}$, there are both global solutions for small initial values of $u_0(x)$ and solutions blowing up in finite time.

It is shown that the number $q_c^* = p - 1 + \frac{p}{N}$ is also a Fujita type critical exponent. When $q = q_c^* = p - 1 + \frac{p}{N}$, in [25] Qi have proved that there exists no global solution of (1.5) with non-negative and continuous initial value, i.e. $q_c^* = p - 1 + \frac{p}{N}$ belongs to the blow-up case.

For more references on this topic, we refer the readers to see [2,18,26] and references therein.

In this paper we shall study the behavior of solutions $u(x, t)$ of (1.5) while the initial values $u_0(x)$ have slow decay near $x = \infty$. For instance, in the case

$$u_0(x) \cong M|x|^{-a}$$

with $M > 0$ and $a \ge 0$, we are interested in the question of global existence and nonexistence of solutions for (1.5) in terms of M and a. These problems have been studied by Lee and Ni [17] and Gui and Wang [12] for the Cauchy problem (1.1). In [16] Huang, Mochizuki and Mukai have obtained similar results for the Cauchy problem of the semilinear system of equations $u_t = \Delta u + v^p$, $v_t = \Delta v + u^q$ with $pq > 1$. Recently, Mukai, Mochizuki, Huang (see [23] for the case $1 < m < p$) and Guo (see [14] for the case $(1 - \frac{2}{N})_+ < m < 1$) have studied the Cauchy problem (1.2). It is shown that for $p > p_m^* = m + \frac{2}{N}$ there is a secondary critical exponent $a^* = \frac{2}{p-m}$ such that the solution of (1.2) blows up in finite time for any initial value $u_0(x)$ which behaves like $|x|^{-a}$ at $|x| = \infty$ if $a \in (0, a^*)$; and there are global solutions for the initial value $u_0(x)$ which behaves like $|x|^{-a}$ at $|x| = \infty$ if $a \in (a^*, N)$.

Motivated by their works, in [19] we have considered the Cauchy problem (1.4) for the case of $N \ge 2$, $p > 1$ and $q > p + 1 + \frac{2}{N}$, and proved that there is a new secondary critical exponent $a^* = \frac{2}{q-p-1}$ such that the solution of (1.4) blows