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Exact solutions for a generalized nonlinear fractional Fokker–Planck equation

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ABSTRACT

This paper is devoted to investigating a generalized nonlinear Fokker–Planck diffusion equation with external force and absorption. We first investigate the integer nonlinear anomalous diffusion, and we obtain the corresponding exact solution expressed by q-exponential function. The solutions of nonlinear diffusion equation with one-fractional derivative and multi-fractional derivative are also studied in detail, and the solutions can have a compact behavior or a long tailed behavior.

1. Introduction

The fractional nonlinear Fokker–Planck-like equations [1–3] have been used to analyze several physical situations that present anomalous diffusion, that usually contain a mix of nonlinear terms and fractional derivatives. In fact, nonlinear diffusion equation and fractional diffusion are successfully applied to several situations due to its wide application in engineering such as frequency-dependent damping behavior of materials, viscoelasticity, diffusion processes etc. A representative nonlinear diffusion equation is $\partial_t c = D\partial^2_{xx}c^\nu$, usually referred to as the *porous medium equation* ([1] and references therein).

Nonlinear diffusion equations in fractal media have been studied by Pascal [4] and Stephenson [5]. Much study [6–8] has been devoted to one-dimensional fractional nonlinear Fokker–Planck equations. Recently, Liang [9] have studied the solutions of a generalized anomalous diffusion equations with fractional derivatives. In order to cover new situations, for instance, the displacement of viscous fluid by less viscous one in a petroleum reservoir and also the fractal or multi-fractal characteristics of porous rocks in which the oil is immersed require a more general approach to take the nonlinear behavior of the interface into account. We propose a generalized fractional nonlinear Fokker–Planck equation

$$\frac{\partial c(x,t)}{\partial t} = -\frac{\partial^{\tau'}}{\partial x^{\tau'}}\left\{D(x)c^m\left[-\frac{\partial^{\tau}}{\partial x^{\tau}}(|x|^{-\mu}c^\nu)\right]^n\right\} - \frac{\partial^\rho}{\partial x^\rho}\{F(x,t)c^\lambda(x,t)\} - \frac{\partial^{\rho'}}{\partial x^{\rho'}}\{A(x,t)c^{\lambda_1}(x,t)\} \tag{1}$$

where $D(x) = D|x|^{-\theta}$ is the anomalous diffusion coefficient, θ characterizes the mass versus radius behavior of a fractal, D is a constant, $c(x,t)$ is the concentration of diffusing material, $F(x,t)$ is an external force applied to the system, the $A(x,t)$ plays the role of an absorbent or source rate related to a reaction process, λ, λ_1, m, n, μ, ν are fitting parameters to be determined experimentally in the nonlinear diffusion model [4], and they are arbitrary real numbers, τ', τ, ρ', ρ are space-fractional derivative operators.

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We use the Riemann–Liouville operator [10–14] for the fractional derivative and we also work with the *positive* x-axis and, later on, we employ symmetry to extend the results to the entire real axis. The mathematical definition of fractional calculus has been the subject of several different approaches, the most frequently encountered definition of a derivative of fractional order is the Riemann–Liouville derivative, in which the fractional order derivative is defined as

$$ {}_0D_t^{\alpha}[f(t)] = \frac{1}{\Gamma(n-\alpha)}\frac{\mathrm{d}^n}{\mathrm{d}t^n}\int_0^t \frac{f(\tau)}{(t-\tau)^{1-n+\alpha}}\mathrm{d}\tau, \tag{2} $$

where $\alpha \in R$ is the order of the operation and n is an integer that satisfies $n-1 \le \alpha < n$, $\Gamma(\cdot)$ is the Gamma function. The solutions of generalized anomalous equations satisfy the sharp initial condition $c(x, 0) = \delta(x)$ and the boundary condition $c(\pm\infty, 0) = 0$ no matter absorption is zero or not.

The remainder of this paper is organized as follows. In Section 2, for the sake of completeness, we briefly consider the case $\tau = \tau' = \rho' = \rho = 1$ in the presence of the external force $F(x, t) = -k_1x + k_{\lambda_2}x|x|^{\lambda_2-1}$, $A(x, t) = -k_2x + k_{\lambda_3}x|x|^{\lambda_3-1}$ for a general time-dependent diffusion coefficient. In Section 3, we study the exact solutions of Eq. (1) for $\tau' = \rho' = \rho = 1$ and arbitrary τ taking the external force $F(x, t) = -V'(x)|x|^{-\gamma}$, $V(x) = B|x|$, and in Section 4 for arbitrary τ', τ, ρ', ρ. In Section 5, we present our conclusions.

2. Integer Fokker–Planck equation

Inserting $F(x, t) = -k_1x + k_{\lambda_2}x|x|^{\lambda_1-1}$, $A(x, t) = -k_2x + k_{\lambda_3}x|x|^{\lambda_3-1}$, $\tau = \tau' = \rho' = \rho = 1$, $\lambda = 1$, $\lambda_1 = 1$ into (1), we have

$$ \frac{\partial c}{\partial t} = -D\frac{\partial}{\partial x}\left\{x^{-\theta}c^m\left[-\frac{\partial(|x|^{-\mu}c^{\nu})}{\partial x}\right]^n\right\} - \frac{\partial}{\partial x}\{(-k_1x + k_{\lambda_2}x|x|^{\lambda_2-1})c\} - \frac{\partial}{\partial x}\{(-k_2x + k_{\lambda_3}x|x|^{\lambda_3-1})c\}. \tag{3} $$

Similarly to the techniques in Refs. [1,2], we restrict our analysis to find the solution of Eq. (3) that can be expressed as a scaled function of the type

$$ c = \frac{1}{\Phi(t)}\tilde{c}(z), \quad z = \frac{x}{\Phi(t)}, \tag{4} $$

where $\Phi(t)$ is a positive time-dependent, by applying Eq. (4) in Eq. (3) and we suppose that $\lambda_1 + m + (\nu + \mu + 1)n + \theta - 1 = 0$, $k_{\lambda_2} = k_{\lambda_3}$, we have

$$ \Phi(t) = \left[\frac{k}{k_1 + k_2}(1 - \mathrm{e}^{-k_1(m+(\nu+\mu+1)n+\theta)t})\right]^{\frac{1}{m+(\nu+\mu+1)n+\theta}}, \tag{5} $$

where we have adopted the solution which satisfies $\Phi(0) = 0$, and the constant k can be obtained from the normalization condition $\int_{-\infty}^{+\infty} c(x, t)\mathrm{d}x = 1$.

It follows from (3) and (5), the solution is given by

$$ \tilde{c} = z^{\frac{\mu}{\nu}} \exp_q\left[\frac{-\sqrt[n]{\bar{k}/D}}{\nu}\int^z \bar{z}^{\frac{\theta\nu-\mu(m-1)}{n\nu}}\left(\bar{z} - \frac{2k_{\lambda_2}}{\bar{k}}\bar{z}^{\lambda_2}\right)^{\frac{1}{n}}\right], \tag{6} $$

where $q = 1 - \nu - \frac{m-1}{n}$, and $\exp_q(x) \equiv (1 + (1-q)x)^{1/(1-q)}$ for $1 + (1-q)x \ge 0$, and $\exp_q(x) \equiv 0$ for $1 + (1-q)x < 0$. Notice that $\exp_q(x)$ is a q-exponential function which emerge from Tsallis formalism by maximizing the Tsallis entropy $S = (1 - \int \rho^q \mathrm{d}x)/(1-q)$ with suitable constraints [15].

3. Fractional Fokker–Planck equation

Inserting external force $F(x, t) = -B|x|^{-\gamma}$, $A(x, t) = a|x|^{-\delta}$ into Eq. (1), we have

$$ \frac{\partial c}{\partial t} = -D\frac{\partial^{\tau'}}{\partial x^{\tau'}}\left\{x^{-\theta}c^m\left[-\frac{\partial^{\tau}(|x|^{-\mu}c^{\nu})}{\partial x^{\tau}}\right]^n\right\} + \frac{\partial^{\rho}}{\partial x^{\rho}}[B|x|^{-\gamma}c^{\lambda}] - \frac{\partial^{\rho'}}{\partial x^{\rho'}}[a|x|^{-\delta}c^{\lambda_1}], \tag{7} $$

where $\theta, \mu, \gamma, \delta \in R$ are parameters, and $D(\neq 0)$, B are arbitrary constants. By applying (4) in the above equation, we obtain

$$ \begin{aligned} -\frac{1}{\Phi^2(t)}\frac{\mathrm{d}\Phi}{\mathrm{d}t}\frac{\mathrm{d}}{\mathrm{d}z}[z\tilde{c}] &= D\frac{1}{\Phi^{m+n(\mu+\nu+\tau)+\theta+\tau'}(t)}\frac{\mathrm{d}^{\tau'}}{\mathrm{d}z^{\tau'}}\left\{z^{-\theta}\tilde{c}^m\left[-\frac{\mathrm{d}^{\tau}}{\mathrm{d}z^{\alpha}}(z^{-\mu}\tilde{c}^{\nu})\right]^n\right\} \\ &\quad + B\frac{1}{\Phi^{\gamma+\lambda+\rho}(t)}\frac{\mathrm{d}^{\rho}}{\mathrm{d}z^{\rho}}\{z^{-\gamma}\tilde{c}^{\lambda}\} - a\frac{1}{\Phi^{\delta+\lambda_1+\rho'}(t)}\frac{\mathrm{d}^{\rho'}}{\mathrm{d}z^{\rho'}}\{z^{-\delta}\tilde{c}^{\lambda_1}\}. \end{aligned} \tag{8} $$