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# Transfer functions for a one-dimensional fluid–poroelastic system subject to an ultrasonic pulse[☆]

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ABSTRACT

A one-dimensional model of an *in vitro* experiment, in which a specimen of cancellous bone is immersed in water and insonified by an ultrasonic pulse, is considered. The modification of the poroelastic model of Biot due to Johnson et al. [D.L. Johnson, J. Koplik, R. Dashen, Theory of dynamic permeability and tortuosity in fluid-saturated porous media, J. Fluid Mech. 176 (1987) 379–402] is used for the cancellous bone segment. By working with series expansions of the Laplace transform in terms of travel-time exponentials, a series of transfer functions for the reflection and transmission of fast and slow waves at the fluid–poroelastic interfaces are derived. The approach obviates numerical solution beyond the discretization involved in the use of the fast Fourier transform.



## 1. Introduction

Cancellous bone consists of a trabecular frame, the interstices of which are filled with blood and fatty marrow. Several authors [1–5] have attempted to model cancellous bone using Biot's model [6–10] for a poroelastic medium. The survey article of Haire and Langton [11] reviews the successes and failures of early attempts to apply the Biot theory to ultrasound propagation through bone. The more recent application of the modification of Biot's model due to Johnson et al. [12] by Fellah et al. [4] and Sebaa et al. [5] have produced good agreement with measured results in the *in vitro* experiment in which a cancellous bone specimen from which the interstitial fatty marrow has been removed is immersed in water and insonified by an ultrasonic pulse.

In this article, it is shown that a simple idea in conjunction with the power of current computer algebra systems can be used to explore the details of transmission and reflection in the *in vitro* experiment. Reflection and transmission coefficients for the Biot model at a fluid–poroelastic interface for a monochromatic time-harmonic sound source were originally worked out by Stoll and Kan [13]. The case of depth-varying parameters was considered by Stern et al. [14]. Wu et al. [15] developed formulas for determining the amounts of energy transferred to the three types of waves arising in a poroelastic medium. Transfer functions for the polychromatic case are derived in [4]. The approach taken in this article will provide some additional capabilities, particularly, that of differentiating among waves with similar arrival times, but different histories of reflection and transmission.

## 2. Modeling the *in vitro* experiment

Consider a poroelastic segment $\mathcal{B}$ occupying $(0, L)$ with fluids $\mathcal{F}_0$ and $\mathcal{F}_1$ occupying $(-\infty, 0)$ and $(L, \infty)$ respectively with a source at $x = x_s < 0$. When the parameters of $\mathcal{F}_0$ and $\mathcal{F}_1$ are the same, this is the model used by Fellah et al. [4]

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to simulate an experiment in which a cancellous bone specimen is immersed in water and insonified by an ultrasonic pulse.

For the poroelastic medium the modification of Biot's model due to Johnson et al. [12] will be used. In this model, the dynamic tortuosity $\alpha(\omega)$ is expressed as a function of the asymptotic tortuosity $\alpha_\infty$, pore fluid viscosity $\eta$, pore fluid density $\rho_f$, permeability $k$, porosity $\beta$, the angular frequency $\omega$ and the viscous characteristic length $\Lambda$

$$\alpha(\omega) = \alpha_\infty \left( 1 + \frac{\eta\beta}{\mathrm{i}\omega\alpha_\infty\rho_f k}\sqrt{1 + \mathrm{i}\frac{4\alpha_\infty^2 k^2 \rho_f \omega}{\eta \Lambda^2 \beta^2}} \right).$$

Let $\hat{g}(s) = L\{f(t)\} = \int_0^\infty \mathrm{e}^{-st} g(t)\mathrm{d}t$ denote the Laplace transform of a function $g$. The fluid–bone system is modeled by the transformed equations [4]

$$\frac{\partial^2 \hat{p}_0}{\partial x^2} - \frac{s^2}{c_0^2}\hat{p}_0 = -\hat{f}(s)\delta(x - x_s), \quad -\infty < x < 0 \tag{1}$$

$$\begin{bmatrix} P & Q \\ Q & R \end{bmatrix} \begin{bmatrix} \dfrac{\mathrm{d}^2\hat{u}}{\mathrm{d}x^2} \\ \dfrac{\mathrm{d}^2\hat{U}}{\mathrm{d}x^2} \end{bmatrix} = s^2 \begin{bmatrix} \tilde{\rho}_{11} & \tilde{\rho}_{12} \\ \tilde{\rho}_{12} & \tilde{\rho}_{22} \end{bmatrix} \begin{bmatrix} \hat{u} \\ \hat{U} \end{bmatrix}, \quad 0 < x < L$$

$$\frac{\partial^2 \hat{p}_1}{\partial x^2} - \frac{s^2}{c_1^2}\hat{p}_1 = 0, \quad L < x < \infty.$$

The quantities $u(x, t)$ and $U(x, t)$ track the motions of the frame and interstitial fluid respectively. For an ultrasonic source $\omega$ is large whence $s \approx \mathrm{i}\omega$ and the coefficients in the high-frequency approximation of Johnson et al. [12] become

$$\tilde{\rho}_{11} = \rho_{11} + \frac{Z}{\sqrt{s}}, \qquad \tilde{\rho}_{12} = \rho_{12} - \frac{Z}{\sqrt{s}}, \qquad \tilde{\rho}_{22} = \rho_{22} + \frac{Z}{\sqrt{s}} \tag{2}$$

where

$$Z = \frac{2\beta\alpha_\infty}{\Lambda}\sqrt{\rho_f \eta}$$

and $\rho_{11}$, $\rho_{12}$, $\rho_{22}$ are the mass-coupling terms in the Biot's model defined in terms of the density of the frame material $\rho_s$, the pore fluid density $\rho_f$, $\beta$, $\alpha_\infty$ and $\omega$

$$\rho_{12} := -\beta\rho_f(\alpha_\infty - 1), \qquad \rho_{22} := \beta\rho_f\alpha_\infty$$
$$\rho_{11} := (1-\beta)\rho_s + \beta\rho_f(\alpha_\infty - 1).$$

The effective elastic constants $P$, $Q$, and $R$ are related to $\beta$, bulk modulus of the pore fluid $K_f$, bulk modulus of the trabecular bone $K_s$, bulk modulus of the porous skeletal frame $K_b$ and the shear modulus of the composite as well as the skeletal frame $G$:

$$P := \frac{(1-\beta)\left(1 - \beta - \frac{K_b}{K_s}\right)K_s + \beta\frac{K_s}{K_f}K_b}{1 - \beta - \frac{K_b}{K_s} + \beta\frac{K_s}{K_f}} + \frac{4}{3}G$$

$$Q := \frac{\left(1 - \beta - \frac{K_b}{K_s}\right)\beta K_s}{1 - \beta - \frac{K_b}{K_s} + \beta\frac{K_s}{K_f}}$$

$$R := \frac{\beta^2 K_s}{1 - \beta - \frac{K_b}{K_s} + \beta\frac{K_s}{K_f}}.$$

The system of equations for the poroelastic segment in (1) can be written

$$\begin{pmatrix} \dfrac{\mathrm{d}^2\hat{u}}{\mathrm{d}x^2} \\ \dfrac{\mathrm{d}^2\hat{U}}{\mathrm{d}x^2} \end{pmatrix} = \left(\frac{s^2}{PR - Q^2}\right) \begin{pmatrix} R\tilde{\rho}_{11} - Q\tilde{\rho}_{12} & R\tilde{\rho}_{12} - Q\tilde{\rho}_{22} \\ -Q\tilde{\rho}_{11} + P\tilde{\rho}_{12} & -Q\tilde{\rho}_{12} + P\tilde{\rho}_{22} \end{pmatrix} \begin{pmatrix} \hat{u} \\ \hat{U} \end{pmatrix}.$$

In terms of the eigenvalues $\lambda_j$ and corresponding eigenvectors $\mathbf{v}_j$, $j = 1, 2$, of the matrix on the right hand side the system becomes

$$\begin{pmatrix} \dfrac{\mathrm{d}^2\hat{u}}{\mathrm{d}x^2} \\ \dfrac{\mathrm{d}^2\hat{U}}{\mathrm{d}x^2} \end{pmatrix} = \frac{s^2}{PR - Q^2}\mathbf{V}\boldsymbol{\Lambda}\mathbf{V}^{-1} \begin{pmatrix} \hat{u} \\ \hat{U} \end{pmatrix}.$$