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Global dynamics of an SEIRS epidemic model with periodic vaccination and seasonal contact rate ☆

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ABSTRACT

Moneim and Greenhalgh [I.A. Moneim, D. Greenhalgh, Use of a periodic vaccination strategy to control the spread of epidemics with seasonally varying contact rate, Math. Biosci. Eng. 2 (2005) 591–611] proposed an SEIRS epidemic model with general periodic vaccination strategy and seasonally varying contact rate. Their investigation shows that when $R_0^{\sup} < 1$, there exists a globally asymptotically stable disease-free periodic state, and when $R_0^{\inf} > 1$, the disease-free solution is unstable and there is at least one positive periodic solution. But they did not find the threshold condition for uniform persistence and extinction of the disease, and left a conjecture—that is, whether the basic reproduction ratio of the time-averaged system can be the threshold parameter or not. The present paper gives a negative answer to this question and provides a thorough global dynamics for this system. Numerical simulations which show our theoretical results are also given.

1. Introduction

Many infectious diseases fluctuate over time and show seasonal patterns of incidence. One of the well-known examples is provided by data on weekly measles notification in England and Wales for the period 1948–1968 (see [1, Fig. 6.3]). Other childhood diseases such as mumps, chickenpox, rubella, and pertussis have also been found to exhibit seasonal behavior. Although not restricted to children, influenza also exhibits distinct seasonality, with higher mortality and morbidity rates in a few months each year. During the first 2–3 years after the first case, the number of human cases of avian influenza also seemed to follow a seasonal pattern [2].

The possibility that the contact rate is periodic has led to the consideration of epidemic models with seasonal forcing [2]. An epidemic SIS model with periodic contact rate was first considered by Hethcote [3]. Epidemic SIR and SEIR models with periodic contact rate were considered by Dietz [4]. More recently Moneim and Greenhalgh [5] studied the following SEIRS model with periodic vaccination in periodic environments:

$$\begin{cases} \dfrac{dS}{dt} = \mu N(1-p) - \beta(t)SI - (\mu + r(t))S + \delta R, \\ \dfrac{dE}{dt} = \beta(t)SI - (\mu+\sigma)E, \\ \dfrac{dI}{dt} = \sigma E - (\mu+\gamma)I, \\ \dfrac{dR}{dt} = \mu N p + r(t)S + \gamma I - (\mu+\delta)R, \end{cases} \tag{1.1}$$

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where $N = S+E+I+R$ is the total population size, and S, E, I and R denote the fractions of population that are susceptible, exposed, infectious and recovered, respectively. $\beta(t)$ is the transmission rate and it is a continuous, positive T-periodic function. p $(0 \leqslant p \leqslant 1)$ is the vaccination rate of all new-born children. $r(t)$ is the vaccination rate of all susceptibles in the population and it is a continuous, positive periodic function with periodic LT, where L is an integer. μ is the common per capita birth and death rate, σ, γ and δ are the per capita rates of leaving the latent stage, infected stage and recovered stage, respectively. It is assumed that parameters μ, δ, σ and γ are positive constants.

Moneim and Greenhalgh [5] established sufficient conditions for uniform persistence and extinction of the disease, but they did not find the threshold between uniform persistence and extinction of the disease, and left an open problem, that is, whether or not a value in the integral form on the basis of the basic reproduction ratio of a time-averaged autonomous system can be the threshold parameter for the persistence and extinction of the disease for (1.1). But our results in this paper will give a negative answer to this problem. And we show that the global dynamics of (1.1) is completely determined by the basic reproduction ratio defined according to the framework given in [6]. Our results really improve the results in [5] in the sense that our condition is a threshold condition between the persistence and extinction of the disease.

This paper is organized as follows. In Sections 2 and 3, we introduce the basic reproduction ratio and show that it acts as a threshold parameter for uniform persistence and global extinction of the disease. In Section 4, we present numerical simulations which demonstrate the theoretical results and we compare with the previous results of [5].

2. Preliminaries and the basic reproduction ratio

Let $(\mathbb{R}^n, \mathbb{R}^n_+)$ be the standard ordered n-dimensional Euclidean space with a norm $\|\cdot\|$. For $u, v \in \mathbb{R}^n$, we write $u \geqslant v$ if $u - v \in \mathbb{R}^n_+$, $u > v$ if $u - v \in \mathbb{R}^n_+ \setminus \{0\}$, and $u \gg v$ if $u - v \in \mathrm{Int}(\mathbb{R}^n_+)$. Let $A(t)$ be a continuous, cooperative, irreducible and LT-periodic $n \times n$ matrix function, $\Phi_A(t)$ be the fundamental solution matrix of the linear ordinary differential system

$$\frac{\mathrm{d}x}{\mathrm{d}t} = A(t)x, \tag{2.1}$$

and $r(\Phi_A(LT))$ be the spectral radius of $\Phi_A(LT)$. By the Perron–Frobenius theorem, $r(\Phi_A(LT))$ is the principal eigenvalue of $\Phi_A(LT)$ in the sense that it is simple and admits an eigenvector $v^* \gg 0$. The following result is useful for our subsequent comparison arguments.

Lemma 2.1 (*See [7, Lemma 2.1]*). *Let $p = \frac{1}{LT} \ln r(\Phi_A(LT))$. Then there exists a positive, LT-periodic function $v(t)$ such that $\mathrm{e}^{pt} v(t)$ is a solution of (2.1).*

It is obvious that any solution of (1.1) with nonnegative initial values is nonnegative. Adding all the equations in (1.1), we have $\frac{\mathrm{d}N}{\mathrm{d}t} = 0$, which has the following implications: the three-dimensional simplex

$$\mathcal{D} := \{(S, E, I, R) \in \mathbb{R}^4_+ : S+E+I+R = N\}$$

is positively invariant; also (1.1) is dissipative and the global attractor is contained in $\mathcal{D}$.

On the simplex $\mathcal{D}$,

$$R(t) = N - S(t) - E(t) - I(t).$$

Thus (1.1) reduces to the following three-dimensional system:

$$\begin{cases} \dfrac{\mathrm{d}S}{\mathrm{d}t} = \mu N(1-p) - \beta(t)SI - (\mu + r(t))S + \delta(N - S - E - I), \\ \dfrac{\mathrm{d}E}{\mathrm{d}t} = \beta(t)SI - (\mu+\sigma)E, \\ \dfrac{\mathrm{d}I}{\mathrm{d}t} = \sigma E - (\mu+\gamma)I. \end{cases} \tag{2.2}$$

The dynamical behavior of (1.1) on $\mathcal{D}$ is equivalent to that of (2.2). Therefore, in the rest of the paper we study the system (2.2) in the region

$$X := \{(S, E, I) \in \mathbb{R}^3_+ : S+E+I \leqslant N\},$$

and formulate our results accordingly.

Next, we show the existence of the disease-free periodic solution of (2.2). To find the disease-free periodic solution of (2.2), we consider

$$\frac{\mathrm{d}S}{\mathrm{d}t} = N(\mu(1-p)+\delta) - (\mu + r(t) + \delta)S, \tag{2.3}$$

with initial condition $S(0) = S^0 \in \mathbb{R}_+$. Clearly, (2.3) admits a unique positive LT-periodic solution

$$\hat{S}(t) = \mathrm{e}^{-\int_0^t (\mu+r(s)+\delta)\mathrm{d}s} \left(\hat{S}(0) + N(\mu(1-p)+\delta) \int_0^t \mathrm{e}^{\int_0^s (\mu+r(\xi)+\delta)\mathrm{d}\xi} \mathrm{d}s \right) \tag{2.4}$$