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Application of the λ-symmetries approach and time independent integral of the modified Emden equation

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ABSTRACT

In this paper we derive the time-independent integral for a nonlinear dissipative system, namely the modified Emden equation, from Lie point symmetries. We employ the recently introduced λ-symmetries method [C. Muriel, J.L. Romero, First integrals, integrating factors and λ-symmetries of second-order differential equations, J. Phys. A: Math. Theor. 42 (2009) 365207–365217] to complete this task. To begin with we recall Lie point symmetries of this system and derive λ-symmetries from the vector fields. The knowledge of λ-symmetries enables us to obtain integrating factors, integrals and the general solution for the linearizable case. While determining the integrating factor from the λ-symmetry for the integrable case we find that this case splits up into three sub-cases. We then obtain the integrating factor and integral for these three sub-cases. The results agree with the ones reported in the literature and thereby give a group theoretical interpretation for the nonstandard time independent integrals exhibited by the system.

1. Introduction

During the past few years attempts have been made to demonstrate several nonlinear dissipative/damped systems do possess nonstandard Lagrangian/Hamiltonian [1–5]. The integrals exhibited by these systems also differ considerably from the conventional conserved quantities. To explore these integrals/general solution one has to employ a versatile algorithm. To do so one may take advantage of the classical Lie algorithm [6–8]. However, many of these nonlinear dynamical systems often lack Lie point symmetries [9,10]. Even when possessing certain Lie point symmetries constructing the integrals from the Lie point symmetries pose yet another challenging problem. Of course, one can try to explore Noether symmetries for these systems and study the connection between symmetries and conserved quantities [6]. However, the Lagrangians admitted by these systems are in general nonstandard ones and consequently the determining equations for the infinitesimal symmetries turn out to be complicated to solve and in certain cases only partial information can be extracted.

To overcome this problem, efforts have been made to generalize the classical Lie algorithm and obtain the integrals and the general solution of nonlinear ordinary differential equations (ODEs). Some of the algorithms developed in the recent literature for this purpose are λ-symmetries [11], telescopical symmetry [12], hidden and non-local symmetries [13], the adjoint symmetry method [14], exponential vector fields [6], and so on. Many of the nonlocal or hidden symmetries can be connected with λ-symmetries [15,16]. The method of finding λ-symmetries for a second order ODE has been discussed

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in depth by Muriel and Romero and the advantage of finding such symmetries has also been demonstrated by them in certain concrete examples [11,15]. Progressing further, in a recent paper, Muriel and Romero, have presented an algorithm to determine integrating factors and integrals from λ-symmetries for a second order ODE [17]. They have shown that once the λ-symmetries for a second order ODE are explored then one can determine the first integral in two different ways. One is by finding the integrating factor μ from λ-symmetry by solving a system of linear partial differential equations for μ (vide Eq. (38)) and then integrating a system of first order partial differential equations for I with known μ and λ (vide Eq. (39)). In the second way one can construct the integral directly from λ-symmetry by a four step algorithm (vide Section 3.2). The underlying integrating factor μ can then be deduced from the integral just by differentiating the latter with respect to $\dot{x}$. Suppose the system under investigation possesses Lie point symmetries then one can derive some of the λ-symmetries from the Lie point symmetries in a direct way [17].

The aim of this paper is to exploit the λ-symmetries approach to the modified Emden equation [1],

$$\ddot{x} = -(\alpha x\dot{x} + \beta x^3), \tag{1}$$

where the over-dot denotes differentiation with respect to t and α and β are arbitrary parameters, and derive nonstandard time independent integrals exhibited by the system for arbitrary values of α and β. Equation (1) has received attention from both mathematicians and physicists for more than a century, see for example Ref. [1,18,19] and references therein. As far as complete integrability is concerned, in a recent work, Chandrasekar et al. [18,19] have shown that the system (1) admits a time independent integral for all values of α and β. This striking result comes out while applying the Prelle–Singer procedure, another tool to obtain integrating factors and integrals for a class of ODEs of any order as well as the system of ODEs [20], to the modified Emden equation. In fact there exists a class of nonlinear dissipative systems that possess nonstandard time independent integrals [4] and the present study helps us to understand the connection between Lie point symmetries and the time independent integrals in these systems as well.

We recall here that Lie's invariance analysis for the Eq. (1) has been carried out by Mahomed and Leach [21] long ago and several subsequent studies have also been made by them and their collaborators on the symmetries exhibited by the system as well as the underlying equation [22–25]. It has been shown that (1) admits eight parameter Lie point symmetries for the choice $\alpha^2 = 9\beta$ and for the choice $\alpha^2 \neq 9\beta$ it possesses only a two parameter Lie point symmetries. In the second case, $\alpha^2 \neq 9\beta$, the system is integrable since it admits a two parameter Lie point symmetries [6–8]. In the first case one gets a linearizable equation. The underlying equation, $\ddot{x}+\alpha x\dot{x}+\frac{\alpha^2}{9}x^3 = 0$, is also referred to as a second order Riccati equation [1]. Very recently multifaceted investigations have been carried out on this equation [26–28] and its generalization. Confining ourselves to the studies where the linearization of this nonlinear ODE alone is considered, it has been shown that it can be linearized by a variety of transformations. For example, it has been shown that the equation, $\ddot{x} + \alpha x\dot{x} + \frac{\alpha^2}{9}x^3 = 0$, can be transformed into a linear equation by (i) invertible point transformation [21], (ii) contact transformation [13] and (iii) nonlocal transformation of different types [13]. Interestingly the linearizable equation (which is also a dissipative system) admits a time independent Hamiltonian $H = -\frac{\alpha}{3}x^2p + (2p)^{\frac{1}{2}}$ with the canonical momentum defined by $p = \frac{1}{2\left(\dot{x}+\frac{\alpha}{3}x^2\right)^2}$ [28]. Here also one can transform this Hamiltonian into a freely falling particle Hamiltonian, $H = \frac{P^2}{2\alpha} + \left(\frac{-\alpha}{3}\right)^{\frac{1}{2}} U$, by using the canonical transformation $x = \frac{3P}{\alpha U}$ and $p = -\frac{\alpha}{6}U^2$.

We note that integrating factors and the first integral of the modified Emden equation with linear external forcing, namely $\ddot{x}+kx\dot{x}+\frac{k^2}{9}x^3+ax = 0$, through the λ-symmetries approach has been reported in Ref. [29]. In this paper the author has found λ-symmetry of the form $\lambda = \frac{\dot{x}}{x} - \frac{k}{3}x$ for the vector field $\frac{\partial}{\partial x}$. From the λ function the author has derived an integrating factor and first integral for this equation. Similarly a connection between λ-symmetries and nonlocal transformation for a class of second order nonlinear ODEs including Liénard type equations has been studied by Muriel and Romero [30]. The authors have shown that among the other equations, the modified Emden equation with linear external forcing can be transformed to the free particle equation, $\frac{d^2F}{d^2G} = 0$, through the nonlocal transformation, $F(t,x) = \phi(t)$, $G(t,x,\dot{x}) = \frac{\sqrt{k}\phi'(t)}{\sqrt{k}t+\arctan\left(\frac{(\dot{x}/x)+kx}{\sqrt{k}}\right)}$, where ϕ is an arbitrary function of t. This transformation is associated with the λ-symmetries equivalent to the canonical pair $(v,\lambda) = (\partial_x, \alpha(t,x)\dot{x} + \beta(t,x))$, where $\alpha = -\left(\frac{F_x}{G}\right)_x\left(\frac{F_x}{G}\right)^{-1}$ and $\beta = -\left(\frac{F_t}{G}\right)_x\left(\frac{F_x}{G}\right)^{-1}$ with F and G given above [30]. However, in this work we discuss the application of the λ-symmetry approach to study the integrability of Eq. (1) only.

We divide our present study into two categories: the linearizable case in the first category and the general case in the second category. Since the Lie point symmetries are known for both the cases we construct λ-symmetries from them and proceed to construct the integral from λ-symmetries in two different routes: In the first route we construct the integrals directly from λ-symmetries using a four step algorithm given in [17]. In the second route we first construct the integrating factors from λ-symmetries and from the integrating factors we obtain the necessary integrals. As far as the linearizable case is concerned to begin with we show that the functions λ_1 and λ_2 (which come out from the vector fields v_2 and v_4) are different by showing the pairs (∂_x, λ_1) and (∂_x, λ_2) are not A-equivalent which in turn guarantees that the first integral associated with functions λ_1 and λ_2 are also functionally independent. From these two integrals we derive the general solution. In the second route also we consider only these two vector fields to construct the integrals. As far as the general case is concerned $(\alpha^2 \neq 9\beta)$ we first consider the time translational symmetry. We then determine the integral in both routes. Interestingly