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Central symmetric solution to the Neumann problem for a time-fractional diffusion-wave equation in a sphere

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ABSTRACT

In this paper, a time-fractional central symmetric diffusion-wave equation is investigated in a sphere. Two types of Neumann boundary condition are considered: the mathematical condition with the prescribed boundary value of the normal derivative and the physical condition with the prescribed boundary value of the matter flux. Several examples of problems are solved using the Laplace integral transform with respect to time and the finite sin-Fourier transform of the special type with respect to the spatial coordinate. Numerical results are illustrated graphically.

1. Introduction

The conventional theory of heat conduction is based on the classical (local) Fourier law, which relates the heat flux vector **q** to the temperature gradient. In combination with a law of conservation of energy, the Fourier law leads to the parabolic heat conduction equation. It is well known that from a mathematical viewpoint, the Fourier law in the theory of heat conduction and the Fick law in the theory of diffusion are identical. In combination with the balance equation for mass, the Fick law leads to the classical diffusion equation. Nonclassical theories, in which the Fourier law and the standard heat conduction equations are replaced by more general equations, constantly attract the attention of researchers. Some of these theories were formulated in terms of the theory of heat conduction, others in terms of the diffusion theory. For an extensive bibliography on this subject and further discussion see, for example, [1–5] and references therein.

The time-fractional diffusion-wave equation can be obtained as a consequence of the balance equation for mass and the time-nonlocal dependence between the matter flux vector **j** and the concentration gradient with the "long-tale" power kernel [6,7] (see also [8])

$$\mathbf{j}(t) = -\frac{k}{\Gamma(\alpha)}\frac{\partial}{\partial t}\int_0^t (t-\tau)^{\alpha-1}\,\mathrm{grad}\,c(\tau)\,\mathrm{d}\tau, \quad 0<\alpha\le 1, \tag{1}$$

$$\mathbf{j}(t) = -\frac{k}{\Gamma(\alpha-1)}\int_0^t (t-\tau)^{\alpha-2}\,\mathrm{grad}\,c(\tau)\,\mathrm{d}\tau, \quad 1<\alpha\le 2, \tag{2}$$

where k is the diffusion conductivity and $\Gamma(\alpha)$ is the Gamma function.

Eqs. (1) and (2) can be interpreted in terms of fractional calculus:

$$\mathbf{j}(t) = -kD_{RL}^{1-\alpha}\,\mathrm{grad}\,c(t), \quad 0<\alpha\le 1, \tag{3}$$

$$\mathbf{j}(t) = -kI^{\alpha-1}\,\mathrm{grad}\,c(t), \quad 1<\alpha\le 2. \tag{4}$$

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Here $I^{\alpha}f(t)$ and $D_{RL}^{\alpha}f(t)$ are the Riemann–Liouville fractional integral and derivative of the order α, respectively. The constitutive Eqs. (3) and (4) yield the time-fractional diffusion-wave equation with Caputo fractional derivative of the order α.

Many of the universal electromagnetic, acoustic, and mechanical responses can be modeled accurately using the fractional diffusion-wave equation [9]. This equation was used [10] to describe diffusion in porous media, which exhibits a fractal geometry, and to study the propagation of mechanical diffusive waves in viscoelastic media with a power-law creep [11]. Based on this equation, Mainardi and Paradisi [12] investigated viscoelastic processes with applications to acoustics and seismology. The time-fractional diffusion-wave equation also describes important physical phenomena in amorphous, colloid and glassy materials, in fractals and percolation clusters, comb structures, dielectrics and semiconductors, biological systems, polymers, random and disordered media and geophysical and geological processes [4–6,13–18].

Starting from the pioneering works [19–21], the time-fractional diffusion-wave equation has received the widespread attention of many researchers. Various problems in curvilinear coordinates were considered in [22–31], among others.

In this paper the solutions to a time-fractional diffusion-wave equation are investigated in a sphere in the case of Neumann boundary conditions. For the first time, two distinct types of Neumann conditions are considered: the mathematical one with the prescribed boundary value of the normal derivative of a function and the physical one with the prescribed boundary value of the matter flux. It should be noted that in the case of classical diffusion ($\alpha = 1$) these two types of boundary condition coincide, but for fractional diffusion-wave equation $\alpha \neq 1$ they are essentially different. The Laplace integral transform with respect to time t allows us to eliminate time-differentiation. In the case of the Dirichlet problem for the time-fractional diffusion-wave equation in a sphere with the prescribed boundary value of a function, the standard substitution $v = rc$ results in the corresponding Dirichlet boundary problem for a finite interval. Therefore, the usual finite sin-Fourier transform with respect to the spatial coordinate r can be applied [23] (see also [32]). In the case of the Neumann problem, the substitution $v = rc$ is inadequate because the boundary condition for the function v becomes more complicated. In this case the finite sin-Fourier transform of the special type [33] should be used.

The results obtained in this paper can be also used as a constituent part for further analysis of nonlinear problems (see, for example, the paper [34] in which the time-fractional diffusion-wave equation with spatial dependent diffusion coefficient was considered).

2. The mathematical Neumann problem

2.1. Statement of the problem

Consider the central symmetric time-fractional diffusion-wave equation in a sphere with a radius R

$$\frac{\partial^{\alpha} c}{\partial t^{\alpha}} = a\left(\frac{\partial^2 c}{\partial r^2} + \frac{2}{r}\frac{\partial c}{\partial r}\right), \quad 0 < r < R,\ 0 < t < \infty,\ 0 < \alpha \le 2, \tag{5}$$

under zero initial conditions

$$t = 0 : c = 0, \quad 0 < \alpha \le 2, \tag{6}$$

$$t = 0 : \frac{\partial c}{\partial t} = 0, \quad 1 < \alpha \le 2, \tag{7}$$

and the prescribed boundary value of the normal derivative

$$r = R : \frac{\partial c}{\partial r} = w(t). \tag{8}$$

In Eq. (5), we use the Caputo fractional derivative [35,36]

$$\frac{\mathrm{d}^{\alpha} f(t)}{\mathrm{d}t^{\alpha}} = \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-\tau)^{n-\alpha-1}\frac{\mathrm{d}^n f(\tau)}{\mathrm{d}\tau^n}\,\mathrm{d}\tau, \quad n-1 < \alpha < n, \tag{9}$$

with the following Laplace transform rule

$$\mathcal{L}\left\{\frac{\mathrm{d}^{\alpha} f(t)}{\mathrm{d}t^{\alpha}}\right\} = s^{\alpha}\mathcal{L}\{f(t)\} - \sum_{k=0}^{n-1} f^{(k)}(0^+)s^{\alpha-1-k}, \quad n-1 < \alpha < n. \tag{10}$$