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Dynamics of transitions in population interactions

Teodoro Lara [a], Jorge Rebaza [b,*]

[a] Departamento de Física y Matemáticas, Universidad de los Andes, N. U. Rafael Rangel, Trujillo, Venezuela

[b] Department of Mathematics, Missouri State University, United States

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A two-species model with transitions between population interactions is studied. Rich dynamics is observed as the number and quality of equilibria change when model parameters and functional responses vary. Existence and stability of equilibria and nonexistence of periodic solutions are established, existence of some bifurcation phenomena are analytically and numerically studied, explicit threshold values are computed to determine the kind of interaction (mutualism, competition, host–parasite) between the species, and several numerical examples are provided to illustrate the main results in this work.

1. Introduction

Two or more species can be found coexisting under different kinds of associations. The static classification of interactions like mutualism and competition among others, is in some cases inadequate as such relation may vary depending on population densities, age or size of individuals, as well as on several environmental parameters. Population models with conditional interaction, represented via nonconstant functions have been the focus of several studies including [1–4]. Besides the rigorous mathematical work, they clearly highlight the practical importance and relevance of studying these models. One important point is that the presence of a species involves cost and benefits to the second species sharing the same environment, and therefore their interspecific relationship needs to be represented by continuous functions that can assume positive and negative values. Nature provides us with several examples for such variable interactions. For instance, ants benefit from their interactions with aphids, because the latter provide certain secretions which are rich in sugars and amino acids; ants at the same time provide protection to aphis from their natural predators. However, the magnitude of these benefits depends on the relative densities of the two populations involved: at low aphid densities, benefits for them are high, but at higher densities such benefits are low, none or even negative [5,1,2]. There are even examples where predator and prey may interchange role, as in the case of lobsters and whelks in the islands of Malgas and Markus, in South Africa, [1,6], all depending again on population densities and environmental parameters.

In this paper, we study the dynamics of a two-species model that incorporates two rational α-functions representing the variable interaction between the species. We study existence and stability of equilibria, possible existence of periodic solutions, bifurcation phenomena, and we explicitly establish threshold values that help determine whether given equilibrium points represent mutualism, competition or host–parasite behavior. Several numerical examples are given to illustrate the main results on the number equilibria and their stability properties, bifurcations, and how the interaction between the species varies with population densities and environmental parameters.

* Corresponding author.
E-mail addresses: tlara@ula.ve (T. Lara), jrebaza@missouristate.edu (J. Rebaza).

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2. Model formulation

We consider the following model

$$\begin{aligned}\frac{dx}{dt} &= x\left(1-\frac{x}{K_1}\right)+\alpha_1(x,y)\frac{xy}{K_1}\\ \frac{dy}{dt} &= y\left(1-\frac{y}{K_2}\right)+\alpha_2(x,y)\frac{xy}{K_2},\end{aligned} \tag{2.1}$$

where $x(t) \geq 0$, $y(t) \geq 0$ represent the population densities, and K_1, K_2 are the carrying capacities of species 1 and 2 respectively. Classically, one considers α_1 and α_2 as fixed parameters, and their signs determine the interspecific interaction. More precisely, we have:

Mutualism, if $\alpha_1 > 0$ and $\alpha_2 > 0$, competition, if $\alpha_1 < 0$ and $\alpha_2 < 0$, host–parasite, if $\alpha_1 > 0$ and $\alpha_2 < 0$, or $\alpha_1 < 0$ and $\alpha_2 > 0$.

Here, we consider functions

$$\alpha_1(y)=\frac{a_1-b_1y}{c_1+d_1y}, \qquad \alpha_2(x)=\frac{a_2-b_2x}{c_2+d_2x}, \tag{2.2}$$

to represent such specific interactions (including saturation effects) between both species, where the parameters $a_{1,2}, b_{1,2}, c_{1,2}, d_{1,2}$, representing changes in environmental conditions are all positive. Thus, the interspecific interactions are not fixed but vary with the environmental parameters and the system state, and can take positive or negative values.

Note. Observe that if $c_1 = c_2 = 1$ and $d_1 = d_2 = 0$, then our model reduces to the one studied in [4], where the authors, among other results, give a detailed analysis of existence and stability of equilibria of the corresponding model.

Invariance and boundedness of solutions. Observe that any solution of (2.1) that starts at $(x_0, 0)$, with $x_0 > 0$ will approach the equilibrium $P_1 = (K_1, 0)$, as dictated by the logistic equation

$$\frac{dx}{dt}=x\left(1-\frac{x}{K_1}\right).$$

Similarly, solutions will approach $P_2 = (0, K_2)$, for any starting point $(0, y_0)$, with $y_0 > 0$. This implies that solutions starting inside the positive quadrant $\mathbb{R}^2_+$ cannot cross the axes, and therefore this region $\mathbb{R}^2_+$ is invariant for the system (2.1).

Where the solutions starting inside the positive quadrant $\mathbb{R}^2_+$ go, depends on the specific interactions represented by the functions α_1 and α_2. For general interaction functions, consider first mutualism, that is, $\alpha_1(x, y) > 0$, and $\alpha_2(x, y) > 0$, for all $x > 0, y > 0$. Then, from (2.1),

$$\frac{dx}{dt} > x\left(1-\frac{x}{K_1}\right) \quad \text{and} \quad \frac{dy}{dt} > y\left(1-\frac{y}{K_2}\right).$$

Recalling that the logistic equation $dw/dt = w(1 - w/k)$, with $w_0 > 0, k > 0$ has strictly positive solutions, and using a differential inequality from [7], we can conclude that $x(t) > w(t) > 0$ and similarly $y(t) > w(t) > 0$ for $t \geq 0$.

With the same reasoning, for the case of competition, i.e. $\alpha_1(x, y) < 0, \alpha_2(x, y) < 0$, we have that $0 < x(t) < w(t) \leq K_1$, if $x_0 < K_1$, and $0 < x(t) < w(t) \leq x_0$, if $x_0 > K_1$, for all $t \geq 0$. Similar inequalities are true for $y(t)$. Both populations stay bounded.

The host–parasite case, say, $\alpha_1(x, y) > 0$ and $\alpha_2(x, y) < 0$, is a combination of the two cases discussed above.

3. Local stability of equilibria

The system (2.1), (2.2) has four equilibria:

$$P_0=(0,0), \qquad P_1=(K_1,0), \qquad P_2=(0,K_2), \qquad P_3=(x_3,y_3),$$

where (x_3, y_3) is the solution of

$$\begin{aligned}K_1-x+\frac{(a_1-b_1y)y}{c_1+d_1y}&=0\\ K_2-y+\frac{(a_2-b_2x)x}{c_2+d_2x}&=0.\end{aligned} \tag{3.1}$$

Denote $A = K_1d_1 + a_1$, and $B = K_2d_2 + a_2$. Then, solving for x in the first equation of the system (3.1), we get $x = \frac{K_1c_1+Ay-b_1y^2}{c_1+d_1y}$. The parabola in the numerator has vertex on the first quadrant, and one positive root

$$\hat{y}=(A+\sqrt{A^2+4b_1c_1K_1})/2b_1. \tag{3.2}$$