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# Modified generalized projective synchronization of a new fractional-order hyperchaotic system and its application to secure communication

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ABSTRACT

This paper presents a new fractional-order hyperchaotic system. The chaotic behaviors of this system in phase portraits are analyzed by the fractional calculus theory and computer simulations. Numerical results have revealed that hyperchaos does exist in the new fractional-order four-dimensional system with order less than 4 and the lowest order to have hyperchaos in this system is 3.664. The existence of two positive Lyapunov exponents further verifies our results. Furthermore, a novel modified generalized projective synchronization (MGPS) for the fractional-order chaotic systems is proposed based on the stability theory of the fractional-order system, where the states of the drive and response systems are asymptotically synchronized up to a desired scaling matrix. The unpredictability of the scaling factors in projective synchronization can additionally enhance the security of communication. Thus MGPS of the new fractional-order hyperchaotic system is applied to secure communication. Computer simulations are done to verify the proposed methods and the numerical results show that the obtained theoretic results are feasible and efficient.



## 1. Introduction

In spite of the 300-year history of fractional calculus [1,2], the applications of fractional calculus to physics, engineering and control processing are just a recent focus of interest [3,4]. It was found that many systems in interdisciplinary fields can be elegantly described with the help of fractional derivatives, for instance, viscoelastic systems [5], electromagnetic waves [6], dielectric polarization [7], quantitative finance [8] and quantum evolution of complex systems [9], and so forth. More examples for fractional-order dynamics can be found in [3] and references therein. These examples and many other similar samples perfectly demonstrate the importance of consideration and analysis of dynamical systems with fractional-order models. There are many material differences between the ordinary differential equation systems (integer-order) and the corresponding fractional-order differential equation systems. Most of the properties or conclusions of the integer-order system cannot be simply extended to the case of the fractional-order one. To date, many fractional-order differential systems such as the fractional-order Rössler system [10], the fractional-order Chen system [11], the fractional-order Lü system [12], the fractional-order unified system [13], etc., display chaotic behavior.

A hyperchaotic system is characterized as a chaotic attractor with more than one positive Lyapunov exponents which can enhance the randomness and higher unpredictability of the corresponding system. So the hyperchaos may be more

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useful in some fields such as communication, encryption etc. Motivated by this, in our work, we numerically investigate the hyperchaotic behaviors of a new fractional-order four-dimensional system. It is found that hyperchaos does exist in the new fractional-order system with order as low as 3.664. The hyperchaotic dynamical behaviors of the system were demonstrated by computer simulations. Numerical evidence shows that the new fractional-order system has two positive Lyapunov exponents.

On the other hand, chaos synchronization has attracted much attention since the seminal work of Pecora and Carroll [14]. Recently, synchronization of fractional-order chaotic systems starts to attract increasing attention due to its potential applications in secure communication and control processing. Various types of synchronization for the fractional-order chaotic systems have been investigated, such as complete synchronization (CS) [15], generalized synchronization (GS) [16], phase synchronization (PhS) [17], anti-synchronization (AS) [18], projective synchronization (PS) [19–21], etc. Amongst all kinds of chaos synchronization, projective synchronization, which was first reported by Mainieri and Rehacek [19], is one of the most noticeable one because it can obtain faster communication with its proportional feature [22,23]. In PS, the responses of the master (drive) and slave (response) systems synchronize up to a constant scaling factor. Recently, Wang and He [24] introduced projective synchronization of the fractional-order chaotic systems via linear separation. Then generalized projective synchronization (GPS) of the fractional-order chaotic systems was studied in [21,25]. However, in the above studies, all the states of the drive and response systems synchronize up to an identical constant scaling factor. In [26], Chen et al. proposed a new hyperchaotic system through adding a nonlinear controller to the third equation of the three-dimensional autonomous Chen–Lee chaotic system. Furthermore, they considered the hybrid projective synchronization (HPS) of the new hyperchaotic systems by using a nonlinear feedback control. But they mainly focus on generalized projective synchronization of the integer-order hyperchaotic system. More recently, Zhou and Zhu [27] investigated the function projective synchronization between fractional-order chaotic systems based on the stability theory of fractional-order systems and tracking control technique. The proposed method can be applied to achieve not only the synchronization between the drive system and the response system with different fractional orders, but also the synchronization between two nonidentical fractional-order chaotic systems.

Motivated by the above discussions, in this paper, we propose a new synchronization phenomenon, modified generalized projective synchronization (MGPS), for a class of fractional-order chaotic systems, where the drive and response systems could be synchronized to a constant scaling matrix. By choosing the scaling factors in the scaling matrix, one can flex the scales of different states independently. The unpredictability of the scaling matrix in MGPS can additionally strengthen the security of communications, which could be employed to get more secure communications. Based on the stability theory of the fractional-order system, the controllers are designed to make the drive and response systems synchronized up to the desired scaling matrix. Moreover, by MGPS, a secure communication scheme is presented. The corresponding numerical simulations have verified the effectiveness of the theoretical results.

This paper is organized as follows. In Section 2, a brief review of the fractional derivative and numerical algorithm for the fractional-order system is given. Dynamics of a new fractional-order hyperchaotic system is numerically studied and demonstrated by computer simulations. In Section 3, a general method of MGPS for coupled fractional-order chaotic systems is presented based on the stability theory of the fractional-order system. In Section 4, MGPS of the new fractional-order hyperchaotic system is derived and numerical simulations show the validity of the proposed synchronization scheme. A chaotic secure communication scheme using MGPS is given in Section 5. Finally, the conclusions of this paper are drawn in Section 6.

## 2. A new fractional-order four-dimensional system

### 2.1. Fractional derivative and its approximation method

There are many definitions for the fractional differential operators [1]. The commonly used definition is the Riemann–Liouville definition, defined by

$$D^{\alpha}x(t) = \frac{d^m}{dt^m} J^{m-\alpha}x(t), \quad \alpha > 0 \tag{1}$$

where $m = \lceil \alpha \rceil$, i.e., $m$ is the first integer which is not less than $\alpha$, $J^{\beta}$ is the $\beta$-order Riemann–Liouville integral operator as described by

$$J^{\beta}z(t) = \frac{1}{\Gamma(\beta)} \int_0^t \frac{z(\tau)}{(t-\tau)^{1-\beta}} d\tau, \quad 0 < \beta \le 1 \tag{2}$$

where $\Gamma(\cdot)$ denotes the gamma function.

Here and throughout, the following definition is applied:

$$D_*^{\alpha}x(t) = J^{m-\alpha}x^{(m)}(t), \quad \alpha > 0 \tag{3}$$

where $m = \lceil \alpha \rceil$. It is common practice to call operator $D_*^{\alpha}$ the Caputo differential operator of order $\alpha$ [28]. The Riemann–Liouville fractional derivative appears unsuitable to be treated by the Laplace transform technique in that,