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# The existence of solutions for drying with coupled phase change in a porous medium

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ABSTRACT

This paper deals with a theoretical mathematical analysis of freezing (desublimation) of moisture in a finite porous medium with a heat flux condition at the boundary. The goal is to generalize [E.A. Santillan Marcus, D.A. Tarzia, Exact solutions for drying with coupled phase-change in a porous medium with a heat flux condition on the surface, Comput. Appl. Math. 22 (2003) 293–311], proving the local existence and uniqueness in time of the solution of this problem. We give the model equations as a free boundary problem, and we prove that the problem is equivalent to a system of Volterra integral equations following the Friedman–Rubinstein's method given in [A. Friedman, Free boundary problems for parabolic equations, I. Melting of solids, J. Math. Mech. 8 (1959) 499–517]. Then, we prove that the problem has a unique local solution in time by using the Banach contraction theorem.



## 1. Introduction

Heat and mass transfer with phase-change problems, taking place in a porous medium, such as evaporation, condensation, freezing, melting, sublimation, and desublimation, have wide application in separation processes, food technology, heat and mixture migration in soils and ground, etc. Due to the nonlinearity of the problem, the solutions usually involve mathematical difficulties. Only a few exact solutions have been found for idealized cases (see [1–6] for example). A large bibliography on free and moving boundary problems for the heat-diffusion equation was given in [7].

Mathematical formulation of the heat and mass transfer in capillary porous bodies has been established by Luikov [8–11]. Two different models were presented by Mikhailov [4] for solving the problem of evaporation of liquid moisture from a porous medium. For the problem of freezing (desublimation) of a humid porous half-space, Mikhailov also presented an exact solution [5]. In these works, it is shown experimentally that the freezing of soil is accompanied by a set of interconnected phenomena in which the migration of moisture towards the freezing front plays an important role. Other problems in this direction are given in [12–16]. Studies in other directions can be found in [17–19].

Nowadays, heat and mass transfer in porous bodies is being studied as an important part of many biological problems (as in [20,21]). Gupta [22] presented an approximate solution to a coupled heat and mass transfer problem involving

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**Nomenclature**

$a_i,\ i=1,2$ Thermal diffusivity of phase $i$
$a_m$ Moisture diffusivity
$c_m$ Specific mass capacity
$c_2$ Specific heat capacity
$k_i,\ i=1,2$ Thermal conductivity of phase $i$
$\rho_m$ Density of moisture
$\tau$ Time
$L$ Latent heat of evaporation of liquid per unit mass
$s(\tau)$ Position of the evaporation front
$t_i, i=1,2$ Temperature of phase $i$
$t_0$ Initial temperature
$t_v$ Temperature at the phase-change state
$u_v$ Constant moisture potential in region 1
$u_0$ Uniform moisture potential
$x$ Space coordinate
$\epsilon$ Coefficient of internal evaporation

*Subscripts*

0 At initial time, $t=0$
1 Dried porous medium, $0<x<s(\tau)$
2 Humid porous medium, $x>s(\tau)$
$v$ At the evaporation front, $x=s(\tau)$

evaporation. The problem Gupta treated has an analytical solution, which is presented in [23]. Heat and mass transfer during drying from an homogeneous point of view are also considered in [24–29]. In [30], exact solutions for the problem of drying with coupled phase change in a porous medium with a heat flux condition at $x=0$ of the type $-q_0/\sqrt{\tau}$, with $q_0>0$, for any value of the Luikov number $L_u$ are obtained. The main goal of our work is to generalize [30] considering a more general prescribed flux at the boundary ($x=0$).

Here, we deal with a theoretical mathematical analysis of freezing (desublimation) of moisture in a finite porous medium with heat flux condition at $x=0$, following [4,16]. In Section 2, we give the model equations as a free boundary problem. The goal of this paper is to prove the local existence and uniqueness in time of the solution of this problem. In Section 3, we prove that the problem is equivalent to a system of Volterra integral equations ((3.20)–(3.23)), following the Friedman–Rubinstein's method given in [31,32] (see also [33–38]). Then, in Section 4, we prove that problem (3.20)–(3.23) has a unique local solution in time by using the Banach contraction theorem.

## 2. Statement of the problem

Let us consider a finite porous medium dried by maintaining a heat flux condition at $x=0$ of the type $-f(t)/k_1$, $f(t)>0$. Initially the body is divided into two regions: the first region is the dried porous medium, $0\le x\le b$, where the temperature $t_1(x,0)$ is given by $\theta(x)$ and the moisture potential $u_1(x,0)$ is $u_v$ (constant), and in the second region (the humid porous medium), $b\le x\le 1$, the temperature $t_2(x,0)$ is given by $t_0(x)$ and the moisture potential $u_2(x,0)$ is $u_0(x)$. The moisture is assumed to evaporate at a constant temperature, with evaporation point $t_v$. The position of the phase-change front at time $\tau$ is given by $x=s(\tau)$, $\tau>0$. It divides the porous body into two regions, and $s(0)=b$. It is also assumed that the moisture potential in the first region, $0<x<s(\tau)$, is constant at $u_v$. It is further assumed that the moisture in vapor form does not take away any appreciable amount of heat from the system. Also, we consider in our model that $a_2\neq a_m$.

Neglecting mass diffusion due to temperature variation, the problem can be expressed as follows.

$$\frac{\partial t_1}{\partial \tau}(x,\tau)=a_1\frac{\partial^2 t_1}{\partial x^2}(x,\tau),\quad 0<x<s(\tau),\ \tau>0 \tag{2.1}$$

$$u_1=u_v,\quad 0<x<s(\tau),\ \tau>0 \tag{2.2}$$

$$\frac{\partial t_2}{\partial \tau}(x,\tau)=a_2\frac{\partial^2 t_2}{\partial x^2}+\frac{\varepsilon L c_m}{c_2}\frac{\partial u_2}{\partial \tau},\quad x>s(\tau),\ \tau>0 \tag{2.3}$$

$$\frac{\partial u_2}{\partial \tau}(x,\tau)=a_m\frac{\partial^2 u_2}{\partial x^2}(x,\tau),\quad x>s(\tau),\ \tau>0. \tag{2.4}$$