



On a Hamiltonian version of a three-dimensional Lotka–Volterra system

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ARTICLE INFO

Article history:

Received 12 September 2011

Accepted 31 January 2012

Keywords:

Hamiltonian dynamics

Lotka–Volterra system

Stability of equilibria

Poincaré compactification

Energy-Casimir mapping

ABSTRACT

In this paper we present some relevant dynamical properties of a three-dimensional Lotka–Volterra system from the Poisson dynamics point of view.

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1. Introduction

The Lotka–Volterra system has been widely investigated in the last years. This system, studied by May and Leonard [1], models the evolution of competition between three species. Among the studied topics related with the Lotka–Volterra system, we recall a few of them together with a partial list of references, namely: integrals and invariant manifolds [2,3], stability [2,4,5], limit cycles [6], cross-diffusion [7], traveling wave [8], analytic behavior [9], delay [10], nonlinear analysis [1], Hamiltonian dynamics [11–15], and many others.

In this paper we consider a special case of the Lotka–Volterra system, recently introduced in [9]. We write the system as a Hamiltonian system of Poisson type in order to analyze the system from the Poisson dynamics point of view. More exactly, in the second section of this paper, we prepare the framework of our study by writing the Lotka–Volterra system as a Hamilton–Poisson system, we find a $SL(2, \mathbb{R})$ parameterized family of Hamilton–Poisson realizations, and also write the system in Lax form. As consequence of the Hamiltonian setting we obtain two new first integrals of the Lotka–Volterra system that generates the first integrals of this system found in [3]. In the third section of the paper we determine the equilibria of the Lotka–Volterra system and then analyze their Lyapunov stability. In the fourth section of the article we present some convexity properties of the image of the energy–Casimir mapping and define some naturally associated semi-algebraic splittings of the image. More precisely, we discuss the relation between the image through the energy–Casimir mapping of the families of equilibria of the Lotka–Volterra system and the canonical Whitney stratifications of the semi-algebraic splittings of the image of the energy–Casimir mapping. In the fifth part of the paper we give a topological classification of the fibers of the energy–Casimir mapping, classification that follows naturally from the stratifications introduced in the above section. Note that in our approach we consider fibers over the regular and also over the singular values of the energy–Casimir mapping. The sixth section is dedicated to the study of the Poincaré compactification of the Lotka–Volterra system. More exactly, we integrate explicitly the Poincaré compactification of the Lotka–Volterra system. For details on Poisson geometry and Hamiltonian dynamics see e.g. [16–19].

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2. Hamilton–Poisson realizations of a 3D Lotka–Volterra system

The Lotka–Volterra system we consider for our study, is governed by the equations:

$$\begin{cases} \dot{x} = -x(x - y - z) \\ \dot{y} = -y(-x + y - z) \\ \dot{z} = -z(-x - y + z). \end{cases} \quad (2.1)$$

Note that the above system is the Lotka–Volterra system studied in [3] in the case $a = b = -1$.

In [3] it is shown that this system admits the following polynomial conservation laws:

$$\begin{aligned} f(x, y, z) &= xyz(x - y)(x - z)(y - z), \\ g(x, y, z) &= x^2y^2 - x^2yz - xy^2z + x^2z^2 - xyz^2 + y^2z^2. \end{aligned}$$

Using a Hamiltonian setting of the problem, we provide two degree-two polynomial conservation laws of the system (2.1) which generates f and g . These conservation laws will be represented by the Hamiltonian and respectively a Casimir function of the Poisson configuration manifold of the system (2.1).

As the purpose of this paper is to study the above system from the Poisson dynamics point of view, the first step in this approach is to give a Hamilton–Poisson realization of the system.

Theorem 2.1. *The dynamics (2.1) has the following Hamilton–Poisson realization:*

$$(\mathbb{R}^3, \Pi_C, H) \quad (2.2)$$

where,

$$\Pi_C(x, y, z) = \begin{bmatrix} 0 & y & x - z \\ -y & 0 & -y \\ -x + z & y & 0 \end{bmatrix}$$

is the Poisson structure generated by the smooth function $C(x, y, z) := -xy + yz$, and the Hamiltonian $H \in C^\infty(\mathbb{R}^3, \mathbb{R})$ is given by $H(x, y, z) := xy - xz$.

Note that, by Poisson structure generated by the smooth function C , we mean the Poisson structure generated by the Poisson bracket $\{f, g\} := \nabla C \cdot (\nabla f \times \nabla g)$, for any smooth functions $f, g \in C^\infty(\mathbb{R}^3, \mathbb{R})$.

Proof. Indeed, we have successively:

$$\Pi_C(x, y, z) \cdot \nabla H(x, y, z) = \begin{bmatrix} 0 & y & x - z \\ -y & 0 & -y \\ -x + z & y & 0 \end{bmatrix} \cdot \begin{bmatrix} y - z \\ x \\ -x \end{bmatrix} = \begin{bmatrix} -x(x - y - z) \\ -y(-x + y - z) \\ -z(-x - y + z) \end{bmatrix} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix},$$

as required. $\square$

Remark 2.2. Since the signature of the quadratic form generated by $C(x, y, z) = -xy + yz$ is $(-1, 0, +1)$, the triple $(\mathbb{R}^3, \Pi_C, H)$ is isomorphic with a Lie–Poisson realization of the Lotka–Volterra system (2.1) on the dual of the semi-direct product between the Lie algebra $\mathfrak{so}(1, 1)$ and $\mathbb{R}^2$.

Remark 2.3. By definition we have that the center of the Poisson algebra $C^\infty(\mathbb{R}^3, \mathbb{R})$ is generated by the Casimir invariant $C(x, y, z) = -xy + yz$.

Remark 2.4. The conservation laws f and g found in [3], are given in terms of the Casimir C and respectively the Hamiltonian H as follows:

$$\begin{aligned} f &= CH(C + H), \\ g &= \frac{1}{2} [C^2 + H^2 + (C + H)^2]. \end{aligned}$$

The next proposition gives other Hamilton–Poisson realizations of the Lotka–Volterra system (2.1).

Proposition 2.5. *The dynamics (2.1) admits a family of Hamilton–Poisson realizations parameterized by the group $SL(2, \mathbb{R})$. More exactly, $(\mathbb{R}^3, \{\cdot, \cdot\}_{a,b}, H_{c,d})$ is a Hamilton–Poisson realization of the dynamics (2.1) where the bracket $\{\cdot, \cdot\}_{a,b}$ is defined by*

$$\{f, g\}_{a,b} := \nabla C_{a,b} \cdot (\nabla f \times \nabla g),$$

for any $f, g \in C^\infty(\mathbb{R}^3, \mathbb{R})$, and the functions $C_{a,b}$ and $H_{c,d}$ are given by:

$$\begin{aligned} C_{a,b}(x, y, z) &= (-a + b)xy + ayz - bxz, \\ H_{c,d}(x, y, z) &= (-c + d)xy + cyz - dxz, \end{aligned}$$

respectively, the matrix of coefficients a, b, c, d is $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL(2, \mathbb{R})$.

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