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A new representation of a proximal subdifferential by employing a directional derivative

An Li

School of Mathematical Sciences, Xiamen University, Xiamen 361005, China

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ABSTRACT

In this paper, a new representation of the proximal subdifferential of a nonsmooth function is presented by using a directional derivative. The upper-semicontinuity property of the proximal subdifferential is proved via the new representation. The existence and necessary conditions of an optimal solution for a class of inf-convolution functions are obtained by using the proximal subdifferential. Finally, a relationship between the proximal subdifferential and the quasidifferential is established. Based on the relation, the proximal subdifferential can be computed easily when the quasidifferential is a polytope.

1. Introduction

In recent years, nonsmooth analysis has increasingly come to play a role in functional analysis, optimization, optimal design, differential equations, and control theory. The proximal subdifferential and the quasidifferential (in the sense of Demyanov and Rubinov) are important tools, used widely in nonsmooth analysis and optimization (see [1–8]). The former cannot be computed easily for it is hard to be tested by its definition; the latter does not admit the upper-semicontinuity property, but it can be computed easily, since it is related closely to the classical directional derivative. It is well known that the proximal subdifferential of a function can be represented by its directional derivative when the function is convex [9]. Our aim is to represent the proximal subdifferential by the directional derivative when the function is only lower semicontinuous, not necessarily convex. The new representation of the proximal subdifferential has versatile applications for dealing with the properties of proximal subdifferentials. We find that the proximal subdifferential possesses the upper-semicontinuity property. We get the existence and necessary conditions of an optimal solution for a class of inf-convolution functions by using the proximal subdifferential. In addition, this makes the calculation of the proximal subdifferential possible when the directional derivative exists.

In Ref. [10], Demyanov introduced the notion of quasidifferential. In the one-dimensional case, a quasidifferential happens to be a pair of nonempty closed intervals. Since then, many publications have discussed quasidifferential properties and the quasidifferential structure (see [11–13]). The subdifferential is very important in nonsmooth optimization and control theory. But it is very difficult to be applied in practice for it is hard to compute. Many researchers have dealt with the relationship between the subdifferential and the quasidifferential. In Ref. [14], the Clark generalized Jacobian can be represented via the quasidifferential. In Ref. [15], Gao presented a method for calculating the proximal subdifferential via the quasidifferential. Motivated by Ref. [15], we try to find a relationship between the proximal subdifferential and the quasidifferential. In our results, the proximal subdifferential can be easily calculated when the quasidifferential is a polytope. Compared with Ref. [15], the proximal subdifferential can be obtained within a less rigid requirement. From two examples, we find that the new representation of the proximal subdifferential is more easy to be tested.

The remainder of this paper is organized as follows. In Section 2, we introduce the notions used throughout the paper and present a new representation of the proximal subdifferential. Upper-semicontinuity and other properties of the proximal

E-mail address: leean1980@163.com.

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subdifferential are proved via the new representation of the proximal subdifferential. In Section 3, we establish a relationship between the proximal subdifferential and the quasidifferential. Based on the relation, the proximal subdifferential can be computed easily when the quasidifferential is a polytope. Finally, an example is given to validate the effectiveness of our results.

2. Representation of the proximal subdifferential

The proximal subdifferential is an important tool used widely in nonsmooth analysis and optimization. But it cannot be computed easily. It is interesting to establish a relationship between the proximal subdifferential and the classical directional derivative that can be computed easily. First, we introduce several notations and concepts which are used widely in nonsmooth analysis. Denote by B, the open unit ball and by $\bar{B}$, the closed unit ball, and set $B(x; r) = \{y \in R^n \mid \|y - x\| < r\}$.

Definition 2.1. $f : R^n \to R$ is lower semicontinuous at x provided that

$$\liminf_{x' \to x} f(x') \geq f(x);$$

f is upper semicontinuous at x provided that

$$\limsup_{x' \to x} f(x') \leq f(x).$$

Definition 2.2. $f : R^n \to R$ be lower semicontinuous. $\zeta \in R^n$ is said to be a proximal subgradient (or P-subgradient) of f at x if there exist positive numbers σ and η such that

$$f(y) \geq f(x) + \langle \zeta, y - x \rangle - \sigma \|y - x\|^2, \quad \forall y \in B(x; \eta).$$

The set of all such ζ is denoted by $\partial_p f(x)$, and is referred to as the proximal subdifferential.

Definition 2.3. A function $f : R^n \to R$ is said to be locally Lipschitz continuous of rank K if, for any $x \in R^n$, there exists a $\rho > 0$ such that

$$\|f(x'') - f(x')\| \leq K \|x'' - x'\|, \quad \forall x'', x' \in B(x; \rho).$$

Definition 2.4. A set-valued function F is said to be upper semicontinuous at x if, for all $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\|x - y\| \leq \delta \Rightarrow F(y) \subset F(x) + \varepsilon B.$$

First, we shall prove the properties of directional derivative.

Lemma 2.1. *Assume that $f : R^n \to R$ is locally Lipschitz continuous of rank K near $x \in R^n$ and the directional derivative $f'(x; v)$ of f at x in every direction $v \in R^n$ exists. Then*

(i) *the function $v \to f'(x; v)$ is positively homogeneous on R^n, and satisfies $|f'(x; v)| \leq K\|v\|$;*

(ii) *$f'(x; v)$ is upper semicontinuous as a function of (x, v) and, as a function of v alone, is locally Lipschitz continuous of rank K on R^n.*

Proof. (i) The fact that $f'(x; \lambda v) = \lambda f'(x; v)$ for any $\lambda \geq 0$ is obvious, so we start by proving the inequality in (i). From the Lipschitz condition we obtain

$$|f'(x; v)| \leq \lim_{t \downarrow 0} \frac{|f(x + tv) - f(x)|}{t} \leq \lim_{t \downarrow 0} \frac{K\|x + tv - x\|}{t} \leq K\|v\|.$$

(ii) Let $\{x_i\}, \{v_i\} \subset R^n$ be sequences such that $x_i \to x$, $v_i \to v$ as $i \to \infty$. Choosing $t_i = K\|x_i - x\|^{\frac{1}{2}} + \frac{1}{i}$, we have

$$\begin{aligned} f'(x_i; v_i) &= \lim_{t \downarrow 0} \frac{f(x_i + tv) - f(x_i)}{t} \leq \lim_{i \to \infty} \frac{f(x_i + t_i v_i) - f(x_i)}{t_i} \\ &\leq \lim_{i \to \infty} \frac{f(x_i + t_i v_i) - f(x_i + t_i v)}{t_i} + \lim_{i \to \infty} \frac{f(x_i + t_i v) - f(x)}{t_i} + \lim_{i \to \infty} \frac{f(x) - f(x_i)}{t_i}. \end{aligned}$$

As $i \to \infty$, we have by the Lipschitz condition that

$$\begin{aligned} \frac{|f(x_i + t_i v_i) - f(x_i + t_i v)|}{t_i} &\leq \frac{K\|x_i - x\| + K\|t_i v_i - t_i v\|}{t_i} \\ &\leq \|x_i - x\|^{\frac{1}{2}} + K\|v_i - v\| \to 0 \end{aligned}$$

and

$$\frac{|f(x) - f(x_i)|}{t_i} \leq \frac{K\|x - x_i\|}{t_i} \leq \|x_i - x\|^{\frac{1}{2}} \to 0.$$