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Bifurcation of peakons and periodic cusp waves for the generalization of the Camassa–Holm equation☆

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ABSTRACT

In this paper, we investigate the generalization of the Camassa–Holm equation

$$u_t + K(u^m)_x - (u^n)_{xxt} = \left[\frac{((u^n)_x)^2}{2} + u^n(u^n)_{xx}\right]_x,$$

where K is a positive constant and $m, n \in \mathbb{N}$. The bifurcation and some explicit expressions of peakons and periodic cusp wave solutions for the equation are obtained by using the bifurcation method and qualitative theory of dynamical systems. Further, in the process of obtaining the bifurcation of phase portraits, we show that $K = \frac{m+n}{1+n} c^{\frac{n-m+1}{n}}$ is the peakon bifurcation parameter value for the equation. From the bifurcation theory, in general, the peakons can be obtained by taking the limit of the corresponding periodic cusp waves. However, we find that in the cases of $n \geq 2, m = n + 1$, when K tends to the corresponding bifurcation parameter value, the periodic cusp waves will no longer converge to the peakons, instead, they will still be the periodic cusp waves. To the best of our knowledge, up until now, this phenomenon has not appeared in any other literature. By further studying the cause of this phenomenon, we show that this planar system has some different characters from the previous Camassa–Holm systems. What is more, we obtain some periodic cusp wave solutions in the form of polynomial functions, which are different from those in the form of exponential functions. Some previous results are extended.

1. Introduction

In 1993, Camassa and Holm [1] derived a shallow water wave equation

$$u_t + 2ku_x - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx}, \tag{1.1}$$

which is called the Camassa–Holm equation or the CH equation. Eq. (1.1) was also derived by Dai [2] as a model equation in hyperelastic rods.

For $k = 0$, Camassa and Holm [1] showed that Eq. (1.1) has solitary waves of the form $u(x, t) = ce^{-|x-ct|}$, which are called peakons due to the discontinuity of the first derivative at the wave peak. For the case of $k \neq 0$ and the wave speed $c = \frac{k}{2}$, Liu and Qian [3] gave three ways, one of which is to take the limit of periodic cusp waves, to seek the peakons of Eq. (1.1). For any parameter k and constant wave speed c, Liu et al. [4] showed that Eq. (1.1) has peakons of the form $u(x, t) = (k + c)e^{-|x-ct|} - k$, which can be seen as weak solutions being similar to those in Refs. [5–7].

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In 2001, Liu and Qian [8] suggested a generalized Camassa–Holm equation

$$u_t + 2ku_x - u_{xxt} + au^m u_x = 2u_x u_{xx} + uu_{xxx}. \tag{1.2}$$

Qian and Tang [9] studied the peakons and periodic cusp waves of Eq. (1.2) when $m = 1$ and showed that periodic cusp waves converge to the corresponding peakons when wave speed c tends to some particular bifurcation parameter value. Tian and Song [10] gave some new peaked solitary wave solutions for Eq. (1.2) when $m = 1, 2, 3$. Khuri [11] gave some explicit expressions of the peakons and discontinuous solitary waves for Eq. (1.2) when $m = 1, 2, 3$.

For the case of $m = 2$, $a = 3$ and $k = 0$, Liu and Ouyang [12] gave a new characteristic of solitary wave solutions, that is, the bell-shaped solitary wave and the peakon coexist for the same wave speed.

In 2003, Guo and Liu [13] studied peaked wave solutions of CH–γ equation

$$u_t + c_0 u_x + 3uu_x - \alpha^2(u_{xxt} + 2u_x u_{xx} + uu_{xxx}) + \gamma u_{xxx} = 0, \tag{1.3}$$

where c_0, γ and $\alpha \neq 0$ are parameters. While in 2004, Tang and Yang [14] further studied the peakons and periodic cusp wave solutions of the CH–γ equation (1.3). In Refs. [13,14], the authors pointed out that the periodic cusp wave solutions converge to the peakons.

From the bifurcation theory and these Refs. [1,3–14], we know that the periodic cusp wave solutions will generally converge to the peakons when some parameters take particular bifurcation parameter values. However, in this paper, we show that sometimes, the periodic cusp waves will no longer converge to the peakons in some special cases, instead, they will still be the periodic cusp waves in the Camassa–Holm type equation. Further, we study the cause of this phenomenon.

Recently, Popivanov et al. [15] introduced the following generalization of the Camassa–Holm equation

$$u_t + K(u^m)_x - (u^n)_{xxt} = \left[\frac{((u^n)_x)^2}{2} + u^n(u^n)_{xx}\right]_x, \tag{1.4}$$

where K is a positive constant and $m, n \in \mathbb{N}$, and dealt with compact traveling waves and peakon type solutions of Eq. (1.4).

In this paper, we employ the bifurcation method and qualitative theory of dynamical systems [3,4,8,9,12–14,16,17] to investigate Eq. (1.4). We obtain the bifurcation of phase portraits for Eq. (1.4). Simultaneously we give some explicit expressions of peakons and periodic cusp wave solutions and reveal some relations between them under some special parameters m, n. What is more, we obtain periodic cusp waves in the form of polynomial functions, while they are appeared in the form of exponential functions in the previous references.

The remainder of this paper is organized as follows. In Section 2, we give the bifurcation of phase portraits. In Section 3, we state our main results about the peakons and periodic cusp waves and the theoretical derivations for the main results. A conclusion will be shown in Section 4.

2. Bifurcation of phase portraits

In this section, we will present the process of obtaining the bifurcation of phase portraits. In this paper, we only consider the cases of $m > n \geq 1$.

For given constant c, substituting $u = \varphi(\xi)$ with $\xi = x - ct$ into Eq. (1.4), it follows that

$$-c\varphi' + K(\varphi^m)' + c(\varphi^n)''' = \left[\frac{((\varphi^n)')^2}{2} + \varphi^n(\varphi^n)''\right]'. \tag{2.1}$$

Note that in this paper, the prime will always stand for derivative with respect to the corresponding variable.

Integrating (2.1) once leads to

$$-c\varphi + K(\varphi^m) + c(\varphi^n)'' = \frac{((\varphi^n)')^2}{2} + \varphi^n(\varphi^n)'', \tag{2.2}$$

where the integral constant is taken as zero.

Letting $\phi(\xi) = \varphi^n(\xi)$, we have $\varphi(\xi) = \phi^{\frac{1}{n}}(\xi)$. It follows from (2.2) that

$$-c\phi^{\frac{1}{n}} + K\phi^{\frac{m}{n}} + c\phi'' = \frac{(\phi')^2}{2} + \phi\phi''. \tag{2.3}$$

Letting $y = \phi'$, we obtain a planar integrable system

$$\begin{cases} \dfrac{d\phi}{d\xi} = y, \\ \dfrac{dy}{d\xi} = \dfrac{K\phi^{\frac{m}{n}} - c\phi^{\frac{1}{n}} - \frac{1}{2}y^2}{\phi - c}, \end{cases} \tag{2.4}$$