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Synchronization of a periodically forced Duffing oscillator with a periodically excited pendulum

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ABSTRACT

In this paper, the analytical conditions for a periodically forced Duffing oscillator synchronized with a chaotic pendulum are developed through the theory of discontinuous dynamical systems. From the analytical conditions, the synchronization invariant domains are developed. For a better understanding of synchronization of two different dynamical systems, the partial and full synchronizations of the Duffing oscillator with the chaotic pendulum are presented for illustrations. The control parameter map is developed from the analytical conditions. Under special parameters, the two systems can be fully and partially synchronized. Since the forced pendulum has librational and rotational chaotic motions, the periodically forced Duffing oscillator can be synchronized only with the librational chaotic motions of the pendulum. It is impossible for the forced Duffing oscillator to be synchronized with the rotational chaotic motions.

1. Introduction

The concept of synchronization goes back to the 17th century. In 1673, Huygens [1] described the synchronization of two pendulum clocks with a weak interaction. In fact, Huygens discussed the synchronization of two modal shapes of vibration. The concepts of present synchronizations concern the synchronizations of two or more systems possessing one or more constraints for synchronicity, and such synchronizations experience the characteristics of asymptotic stability. Once the two or more systems form a state of synchronization for a specific constraint, such a state should be stable (see, e.g., [2,3]). The concept of modern synchronization of two dynamical systems was introduced by Pecora and Carroll [4] in 1989, and they presented a criterion of the sub-Lyapunov exponents to determine the synchronization of two systems connected with common signals. The common signals act as constraints for such systems. According to this idea, the synchronized circuits for chaos were presented by Carroll and Pecora [5].

Since then, the research has focused on developing the corresponding control methods and schemes to achieve the synchronization of two dynamical systems with constraints. For instance, in 1992, Pyragas [6] presented two methods for chaos control with a small time continuous perturbation, to achieve a synchronization of two chaotic dynamical systems. In 1994, Kapitaniak [7] used such a continuous control to present the synchronization of two chaotic systems. Ding and Ott [8] pointed out that the slave system (receiver system) does not necessarily have to be a replica of part of the master system. In 1995, Rulkov et al. [9] discussed a generalized synchronization of chaos in directionally coupled chaotic systems. Kocarev and Parlitz [10] developed a general method to construct chaotic synchronized systems, which decomposes the given systems into active and passive systems. In 1996, Peng et al. [11] presented the chaotic synchronization of n-dimensional systems, and Pyragas [12] discussed the weak and strong synchronizations of chaos by the coupling strength

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of two dynamical systems. In 1997, Ding et al. [13] provided a review on the control and synchronization of chaos in high-dimensional dynamical systems. In addition, Boccaletti et al. [14] presented an adaptive synchronization of chaos for secure communication. Abarbanel et al. [15] used a small force to control a dynamical system to given orbits. In 1998, Pyragas [16] systematically introduced some basic ideas about the generalized synchronization of chaos. In 1999, Yang and Chua [17] used linear transformations to investigate generalized synchronization. In 2004, Campos and Urias [18] presented a mathematical description of multi-modal synchronization with chaos. The definition of master–slave synchronization was given, and a multivalued, synchronized function was introduced. Koronovskii et al. [19] discussed the duration of a process of complete synchronization of two coupled, identical chaotic systems. In 2006, Teufel et al. [20] presented the synchronization of two flow-excited pendula, which can recall Huygens' work [1]. In 2002, Boccaletti et al. [21] gave a systematical review of the synchronization of chaotic systems. The definitions and concepts are further clarified. In 2006, Chen et al. [22] gave a review on stability of synchronized dynamics and pattern formation in coupled systems. In addition, there has been interest in the synchronization of discrete systems with mappings. In 1997, Pecora et al. [23] discussed volume-preserving and volume-expanding synchronized chaotic systems through discrete maps. Stojanovski et al. [24] used symbolic dynamics to investigate chaos synchronization, and information entropy was introduced to the synchronization of chaotic systems through discrete maps. In 2001, Rulkov [25] discussed a regularization of synchronized chaotic bursts. Further, Afraimovich et al. [26] gave a mathematical investigation of the generalized synchronization of chaos in non-invertible maps in 2002. In 2003, Barreto et al. [27] discussed the geometrical behavior of chaos synchronization through discrete maps.

The synchronization of two or more dynamical systems means that the corresponding flows of the two or more dynamical systems are constrained under specific constraint conditions for a time interval. If the constraint conditions are treated as constraint boundaries, the synchronization of the two or more dynamical systems can be investigated by the theory of discontinuous dynamical systems. In 2005, Luo [28] developed a theory for discontinuous dynamical systems (see also [29, 30]). In 2009, Luo [31] used the theory of discontinuous dynamical systems to develop a theory for synchronization of dynamical systems with specific constraints. In this paper, such a theory for dynamical system synchronization will be used to investigate the synchronizations of two completely different dynamical systems. Usually, one has investigated the synchronization of two similar dynamical systems, and the two similar dynamical systems can be simplified to create an error dynamical system, in which the Lyapunov method can be used to determine the asymptotical stability. In fact, using the new theory of synchronization theory, the two dynamical systems need not be similar. Thus, in this paper, we will consider that a periodically forced Duffing oscillator will be synchronized with a chaotic pendulum. Consider the chaotic pendulum to be the master system and the periodically forced Duffing oscillator to be the slave system. Under feedback control, it is investigated how the periodically Duffing oscillator will be synchronized with the chaotic pendulum. The partial and full synchronizations of the two systems will be discussed, and the analytical conditions for the two-system synchronization will be developed. The switching scenarios between asynchronized and synchronized states of the two systems will be presented, and the parameter map for such synchronization will be developed. To help one understand the synchronizations, partial and full synchronizations will be illustrated through the velocity responses, phase planes and analytical conditions.

2. Master and slave systems

Consider a periodically driven pendulum as a master system:

$$\ddot{x} + a_0 \sin x = A_0 \cos \omega t. \tag{1}$$

Consider a periodically forced, damped Duffing oscillator as a slave system:

$$\ddot{y} + d_1\dot{y} - a_1 y + a_2 y^3 = Q_0 \cos \Omega t. \tag{2}$$

To enforce the slave system of the Duffing oscillator to synchronize with the master system of the pendulum, a control law should be exerted in the slave system. The following state variables for the slave and master systems are introduced

$$\mathbf{x} = (x_1, x_2)^{\mathrm{T}} \quad \text{and} \quad \mathbf{y} = (y_1, y_2)^{\mathrm{T}}, \tag{3}$$

and the corresponding vector fields are defined as

$$\mathscr{F}(\mathbf{x}, t) = (x_2, \mathscr{F}_2(\mathbf{x}, t))^{\mathrm{T}} \quad \text{and} \quad \mathbf{F}(\mathbf{y}, t) = (y_2, F_2(\mathbf{y}, t))^{\mathrm{T}}. \tag{4}$$

So, the master system is in the form

$$\dot{\mathbf{x}} = \mathscr{F}(\mathbf{x}, t), \tag{5}$$

where

$$\dot{x}_1 \equiv x_2 \quad \text{and} \quad \mathscr{F}(\mathbf{x}, t) = -a_0 \sin x_1 + A_0 \cos \omega t. \tag{6}$$

The slave system becomes

$$\dot{\mathbf{y}} = \mathbf{F}(\mathbf{y}, t), \tag{7}$$