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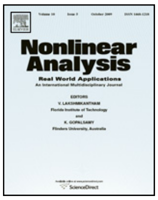

Dynamics of a single species under periodic habitat fluctuations and Allee effect

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ABSTRACT

The dynamics of a single species that goes extinct when rare is described by a nonlinear differential equation

$$N' = rN\left(1 - \frac{N}{K}\right)\left(\frac{N}{K} - \frac{A}{K}\right),$$

where a parameter A $(0 < A < K)$ is associated with the Allee effect, r is the intrinsic growth rate and K is the carrying capacity of the environment. The purpose of this paper is to study the existence of periodic solutions and their stability properties assuming that r, A and K are continuous T-periodic functions. Using rather elementary techniques, we completely describe population dynamics analyzing influences of both strong $(A > 0)$ and weak $(A < 0)$ Allee effects. Thus, we answer questions regarding the location of positive periodic solutions and their stability complementing the research in a recent paper by Padhi et al. [Seshadev Padhi, P.D.N. Srinivasu, G. Kiran Kumar, Periodic solutions for an equation governing dynamics of a renewable resource subjected to Allee effects, Nonlinear Anal. RWA 11 (2010) 2610–2618]. Bounds for periodic solutions and estimates for backward blow-up times are also established. Furthermore, we demonstrate advantages of our approach on a simple example to which the results in the cited paper fail to apply.

1. Introduction

The dynamics of a single non-structured population is directly influenced by regular changes in environmental conditions such as climate, food availability, predator scarcity, etc. Therefore, seasonal habitat fluctuations should be preferably taken into consideration in mathematical models due to the significant effect they have on the population density, even during brief periods when the physical and biological environments remain nearly constant. In fact, Nicholson [1] argued that "any periodic change of climate tends to impose its period upon oscillations of internal origin or to cause such oscillations to have a harmonic relation to periodic climate changes." Many researchers emphasized particular importance of periodic and almost periodic fluctuations in mathematical biology. For instance, Vance and Coddington [2] pointed out that "periodic time variation holds considerable promise as a means to explore time-varying ecological processes."

Although rapid progress in mathematical biology within the last few decades led to incorporation of time-varying parameters in many models, the effect of environmental fluctuations is still being quite often underestimated or even neglected. Henson and Cushing [3, pp. 201–202] stressed that "fluctuating environments are of particular interest to population biologists. Despite this fact, the vast majority of mathematical models used in population dynamics and ecology are autonomous and assume a constant environment. As a result, virtually all fundamental principles in theoretical

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population dynamics are based upon the assumption of a constant environment: monotonic logistic growth, competitive exclusion and ecological niche, predator–prey oscillations, and so on.'' In the cited paper, the authors obtained very interesting results that ''provide the first rigorous evidence, via model analysis of laboratory data, that an effective periodicity can have a positive effect on total population biomass.'' This positive development is in line with the prediction made by Brauer and Sánchez [4, p. 243] who emphasized that ''a general theory of the qualitative behavior of periodic population models, both single species and interacting species, would have many applications.'' Development of such a theory appears to be not an easy task, as the discussion for a relatively simple logistic model in [5] demonstrates, see also the references cited therein.

In this paper, we are concerned with a nonlinear differential equation

$$\dot{N}(t) = r(t) N(t) \left(1 - \frac{N(t)}{K(t)}\right) \left(\frac{N(t)}{K(t)} - \frac{A(t)}{K(t)}\right) \tag{1}$$

with continuous, positive T-periodic functions $r(t)$, $K(t)$ and a continuous T-periodic function $A(t)$. Eq. (1) describes the dynamics of a single species subject to the Allee effect, cf. [6] where the case of constant coefficients is dealt with. This equation can be also written in a compact form

$$\dot{N}(t) = N(t) g(N(t)), \tag{2}$$

where

$$g(N(t)) = r(t) \left(1 - \frac{N(t)}{K(t)}\right) \left(\frac{N(t)}{K(t)} - \frac{A(t)}{K(t)}\right).$$

In Eqs. (1) and (2), N denotes the population size, the function $g(N)$ stands for the density-dependent per capita growth rate, $r(t)$ denotes the maximum per capita population growth rate without the Allee effect, $A(t)$ is the Allee threshold, that is, a critical population size or density below which the per capita population growth rate becomes negative, $K(t)$ is the carrying capacity of the environment.

A so called Allee effect occurs when positive density dependence dominates at low population size; for $A > 0$ (strong Allee effect), it characterizes the dynamics of a single population that goes extinct when rare. This effect is often caused by difficulties in mate finding; it may also depend, as indicated by Lewis and Kareiva [7], on other factors such as inbreeding depression, food exploitation, predator avoidance of defense, etc. Recent studies indicate that a strong Alee effect ($A > 0$) can give rise to a complex dynamics even in simple models arising in mathematical ecology and epidemiology; it can lead to critical population size or population levels below which the population crashes to extinction. Courchamp et al. [8] pointed out that ''studies demonstrating Allee effects and determining their causal mechanisms, either theoretically or empirically, ought to be more numerous in the future.'' For more information on the wide variety of Allee effects in mathematical ecology, we refer to an excellent monograph by Courchamp et al. [9] and review papers by Boukal and Berec [10] or Berec et al. [11].

Research reported in this paper has been strongly motivated by a very recent contribution due to Padhi et al. [12] who discussed existence of periodic solutions to a general scalar differential equation

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = -A(t)x(t) + f(t, x(t)), \tag{3}$$

where A and f are continuous T-periodic real valued functions on $\mathbb{R}$ and $\mathbb{R}^2$ respectively. Using the Legget–Williams multiple fixed point theorem [13], Padhi et al. [12, Theorem 3, Corollary 4] established existence of at least two positive periodic solutions to Eq. (3). As an application, a sufficient condition for the existence of periodic solutions to equation

$$\frac{\mathrm{d}y(t)}{\mathrm{d}t} = a(t)y(t)(y(t) - b(t))(c(t) - y(t)) \tag{4}$$

describing the dynamics of a single species subject to the Allee effect has been derived. In Eq. (4), all coefficients are positive T-periodic functions; it is also assumed that $b(t) < c(t)$, for all $t \in \mathbb{R}$. For the convenience of the reader, we formulate one of the principal results obtained in the cited paper.

Theorem 1 *([12, Theorem 5]). Let*

$$M = \int_0^T a(s)c^2(s)\,\mathrm{d}s \quad \text{and} \quad L = \int_0^T a(s)b(s)c(s)\,\mathrm{d}s. \tag{5}$$

If

$$\frac{M + L + \sqrt{(M+L)^2 - 4M\exp(-L)(\exp(L) - 1)}}{2M} > \frac{\exp(2L) - \exp(-L)}{M + L}, \tag{6}$$

Eq. (4) has at least two positive T-periodic solutions.