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Homoclinic solutions for a kind of neutral differential systems

Bo Du *

Department of Mathematics, Nanjing Normal University, Nanjing, Jiangsu, 210097, PR China
Department of Mathematics, Huaiyin Normal University, Huaian, Jiangsu, 223300, PR China

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ABSTRACT

Employing an extension of Mawhin's continuation theorem and some analysis methods, we obtain the existence of homoclinic orbits for a type of neutral differential systems

$$(u(t) - Cu(t-\tau))'' + \frac{d}{dt}\nabla F(u(t)) + \nabla G(u(t)) = e(t),$$

where $F \in C^2(\mathbb{R}^n, \mathbb{R})$; $G \in C^1(\mathbb{R}^n, \mathbb{R})$; $e \in C(\mathbb{R}, \mathbb{R}^n)$; $C = diag(c_1, c_2, \ldots, c_n)$; c_i and τ are given constants, $i = 1, 2, \ldots, n$.

1. Introduction

This paper is devoted to investigating the existence of homoclinic solutions for a kind of neutral differential systems as follows:

$$(u(t) - Cu(t-\tau))'' + \frac{d}{dt}\nabla F(u(t)) + \nabla G(u(t)) = e(t), \tag{1.1}$$

where $F \in C^2(\mathbb{R}^n, \mathbb{R})$; $G \in C^1(\mathbb{R}^n, \mathbb{R})$; $e \in C(\mathbb{R}, \mathbb{R}^n)$; $C = diag(c_1, c_2, \ldots, c_n)$; c_i $(i = 1, 2, \ldots, n)$ and τ are given constants.

As is well known, a solution $u(t)$ of system (1.1) is named homoclinic (to $\mathbf{0} = (0, 0, \ldots, 0)$) if $u(t) \to \mathbf{0}$ and $u'(t) \to \mathbf{0}$ as $|t| \to +\infty$. In addition, if $u \neq \mathbf{0}$, then u is called a nontrivial homoclinic solution.

For the case $C = \mathbf{0}$, system (1.1) transforms into a classic second-order Hamiltonian system

$$u''(t) = \nabla F(t, u(t)) + f(t) \tag{1.2}$$

which has been extensively studied by many authors. For example, Lzydorek and Janczewska [1] obtained the existence of homoclinic solutions for (1.2). After that, Tang and Xiao [2] generalized the results of [1] by using a different approach. In recent papers, we note that many authors studied the following second-order systems with p-Laplacian

$$\frac{d}{dt}(|u'(t)|^{p-2}u'(t)) = \nabla F(t, u(t)) + f(t), \tag{1.3}$$

and obtained the existence of homoclinic solutions for (1.3), for more details, we refer the reader to see [3,4] for work on this subject. The existence of homoclinic orbits is one of the most important problems in the theory of Hamiltonian systems. In the past decades, many authors investigated homoclinic orbits for Hamiltonian systems by using critical point theory. One can refer to [5–8] and the references therein for detailed discussions.

* Correspondence to: Department of Mathematics, Nanjing Normal University, Nanjing, Jiangsu, 210097, PR China. Tel.: +86 15061217240.
E-mail address: dubo7307@163.com.

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Like in the work of Rabinowitz in [6], Lzydorek and Janczewska in [9], the existence of a homoclinic solution for the system is obtained as a limit of a certain sequence of $2kT$-periodic solutions for the following system:

$$(u(t) - Cu(t-\tau))'' + \frac{d}{dt}\nabla F(u(t)) + \nabla G(u(t)) = e_k(t), \tag{1.4}$$

where $k \in \mathbb{N},\ e_k : \mathbb{R} \to \mathbb{R}^n$ is a $2kT$-periodic function such that

$$e_k(t) = \begin{cases} e(t), & t \in [-kT, kT - \varepsilon_0), \\ e(kT - \varepsilon_0), & t \in [kT - \varepsilon_0, kT], \end{cases} \tag{1.5}$$

$\varepsilon_0 \in (0, T)$ is a constant independent of k. But to the authors' knowledge, there are few papers discussing homoclinic solutions for neutral differential systems. Furthermore, in our approach, the existence of $2kT$-periodic solutions to system (1.4) is obtained by using an extension of Mawhin's continuation theorem [10], not by using the methods of critical point theory, which is quite different from the approach of [1–5,8,6].

2. Preliminaries

Throughout the paper, $\langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ denotes the standard inner product and $|\cdot|$ is the induced Euclidean norm on $\mathbb{R}^n$. For each $k \in \mathbb{N}$, denote

$$C_{2kT} = \{x | x \in C(\mathbb{R}, \mathbb{R}^n),\ x(t+2kT) \equiv x(t)\}, \qquad C^1_{2kT} = \{x | x \in C^1(\mathbb{R}, \mathbb{R}^n),\ x(t+2kT) \equiv x(t)\}$$

and $|x|_0 = \max_{t\in[0,2kT]} |x(t)|$. If the norms C_{2kT} and C^1_{2kT} are defined by $\|\cdot\|_{C_{2kT}} = |\cdot|_0$ and $\|\cdot\|_{C^1_{2kT}} = \max\{|x|_0, |x'|_0\}$, respectively, then C_{2kT} and C^1_{2kT} are both Banach spaces. Furthermore, for $\varphi \in C_{2kT},\ \|\varphi\|_r = (\int_{-kT}^{kT} |\varphi(t)|^r dt)^{1/r},\ r > 1$. Define linear operator:

$$A : C_T \to C_T, \quad [Ax](t) = x(t) - cx(t-\tau), \quad \forall t, c \in \mathbb{R}.$$

Lemma 2.1 ([11,12]). *If $|c| \neq 1$, then operator A has continuous inverse A^{-1} on C_T, satisfying*

$$[A^{-1}f](t) = \begin{cases} \sum_{j\geq 0} c^j f(t - j\tau), \ |c| < 1, & \forall f \in C_T, \\ -\sum_{j\geq 1} c^j f(t + j\tau), \ |c| > 1, & \forall f \in C_T \end{cases}$$

and

$$\|A^{-1}f\| \leq \frac{\|f\|}{|1 - |c||}, \quad \forall f \in C_T.$$

In order to study the existence of $2kT$-periodic solutions for system (1.4), for each $k \in \mathbb{N}$, by (1.5), $e_k \in C_{2kT}$. Let $X_k = C^1_{2kT}$. We need the following lemmas.

Lemma 2.2 ([10]). *Suppose that Ω is an open bounded set in X_k such that the following conditions hold:*

(1) *For each $\lambda \in (0, 1)$, the system*

$$(u(t) - Cu(t-\tau))'' + \lambda \frac{d}{dt}\nabla F(u(t)) + \lambda \nabla G(u(t)) = \lambda e_k(t)$$

has no solution on $\partial\Omega$.

(2) *The system*

$$\Delta(a) = \frac{1}{2kT}\int_{-kT}^{kT} [\nabla G(a) - e_k(t)]dt = \mathbf{0}$$

has no solution on $\partial\Omega \cap \mathbb{R}^n$.

(3) *The Brouwer degree $d_B\{\Delta, \Omega \cap \mathbb{R}^n, \mathbf{0}\} \neq 0$.*

Then system (1.4) has a $2kT$-periodic solution in $\bar{\Omega}$.

Lemma 2.3. *If $u : \mathbb{R} \to \mathbb{R}^n$ is continuously differentiable on $\mathbb{R},\ a > 0,\ \mu > 1$ and $p > 1$ are constants, then for every $t \in \mathbb{R}$, the following inequality holds:*

$$|u(t)| \leq (2a)^{-\frac{1}{\mu}} \left(\int_{t-a}^{t+a} |u(s)|^\mu ds\right)^{\frac{1}{\mu}} + a(2a)^{-\frac{1}{p}} \left(\int_{t-a}^{t+a} |u'(s)|^p ds\right)^{\frac{1}{p}}.$$

This lemma is a special case of Lemma 2.2 in [3].