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Homoclinic solutions for a class of second-order Hamiltonian systems[☆]

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ABSTRACT

A new result for the existence of homoclinic orbits is obtained for the second-order Hamiltonian systems $\ddot{u}(t)+\nabla V(t,u(t))=f(t)$, where $t\in R, u\in R^n$ and $V\in C^1(R\times R^n,R)$, $V(t,x)=-K(t,x)+W(t,x)$ is T-periodic with respect to t, $T>0$ and $f:R\to R^n$ is a continuous and bounded function. This result generalizes and improves some existing results in the known literature.

1. Introduction and main results

In this paper, we shall study the existence of homoclinic orbits for the following second-order Hamiltonian systems

$$\ddot{u}(t)+\nabla V(t,u(t))=f(t), \tag{HS}$$

where $t\in R, u\in R^n, f:R\to R^n$ and $V:R\times R^n\to R$. As usual, we say that a solution $u(t)$ of (HS) is nontrivial homoclinic (to 0) if $u\not\equiv 0$, $u(t)\to 0$ and $\dot{u}(t)\to 0$ as $t\to\pm\infty$.

Recently, the existence and multiplicity of periodic solutions and homoclinic orbits for system (HS) have been studied extensively via critical point theory (see [1–24]). Most of them deal with the superquadratic case (see [1–4,6–16,19–21]) and [5,17,22–24] deal with the subquadratic case. Moreover, many evolution processes are characterized by the fact that at certain moments of time they experience a change of state abruptly; thus impulsive differential equations appear as a natural description of observed evolution phenomena of several real world problems. Due to their applications in many fields, second-order Hamiltonian systems with impulses via critical point theory have been recently considered in [25,26], and in [27], Tian et al. studied some boundary value problems for impulsive differential equations by variational approach.

In recent paper [9], Izydorek and Janczewska proved the following theorem.

Theorem A *(See [9]). Assume that V and f satisfy the following conditions:*

(V1) *$V(t,x)=-K(t,x)+W(t,x)$, where $K,W:R\times R^n\to R$ are C^1-maps, T-periodic with respect to t, $T>0$;*
(V2) *there are constants $b_1,b_2>0$ such that for all $(t,x)\in R\times R^n$*

$$b_1|x|^2\le K(t,x)\le b_2|x|^2;$$

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(V3) *for all* $(t, x) \in R \times R^n, K(t, x) \le (\nabla K(t, x), x) \le 2K(t, x)$;
(V4) $\nabla W(t, x) = o(|x|)$, *as* $|x| \to 0$ *uniformly with respect to* t;
(V5) *there is a constant* $\mu > 2$ *such that for every* $t \in R$ *and* $x \in R^n \setminus \{0\}$

$$0 < \mu W(t, x) \le (\nabla W(t, x), x);$$

(V6) $f : R \to R^n$ *is a continuous and bounded function;*
(V7) *let* $\bar{b}_1 = \min\{1, 2b_1\} > 2M$, $0 < \beta < \bar{b}_1 - 2M$ *and*

$$\int_R |f(t)|^2 dt \le \left(\frac{\beta}{2C^*}\right)^2,$$

where

$$M = \sup\{W(t, x) | t \in [0, T], x \in R^n, |x| = 1\}, \tag{1.1}$$

and C^* *is a positive constant determined by* (1) *in* [9] *and depends upon* T. *When* $T \ge \frac{1}{2}$, $C^* = \sqrt{2}$.
Then system (HS) possesses a nontrivial homoclinic solution.

We notice that the specific form of C^* is not given when $0 < T < \frac{1}{2}$ in [9] and even if $f = 0$, it is still possible that condition (V7) is not satisfied by (HS). Therefore, Theorem A does not generalize completely previous results such as [15] in this sense. Motivated by papers [9,15], in this paper, we will obtain a new criterion for guaranteeing that (HS) has one nontrivial homoclinic solution by using more general conditions, especially, $W(t, x)$ satisfies a kind of new superquadratic condition which is different from the corresponding condition in the known literature. The main results are the following theorems.

Theorem 1.1. *Assume that* V *and* f *satisfy assumptions* (V1), (V3), (V6) *and the following conditions:*

(H1) *there is a constant* $b > 0$ *such that*

$$K(t, 0) = 0, \qquad K(t, x) \ge b|x|^2, \quad \text{for all } (t, x) \in R \times R^n.$$

(H2) $W(t, 0) \equiv 0$ *and* $\nabla W(t, x) = o(|x|)$, *as* $|x| \to 0$ *uniformly with respect to* t;
(H3) *there are two constants* $\mu > 2$ *and* $\nu \in [0, \mu - 2)$ *such that*

$$0 < \mu W(t, x) \le (\nabla W(t, x), x) + \nu b|x|^2, \quad \text{for all } (t, x) \in R \times R^n \setminus \{0\};$$

(H4)

$$\liminf_{|x|\to\infty} \frac{W(t, x)}{|x|^2} > \frac{\pi^2}{2T^2} + m_1,$$

uniformly with respect to t, *where*

$$m_1 = \sup\{K(t, x) | t \in [0, T], \ x \in R^n, |x| = 1\}; \tag{1.2}$$

(H5)

$$\int_R |f(t)|^2 dt < \frac{1}{C^2}\left(\min\left\{\frac{\delta}{2}, \left(1 - \frac{\nu}{\mu - 2}\right) b\delta - M\delta^{\mu-1}\right\}\right)^2,$$

where M *is determined by* (1.1), $C = \sqrt{2}(1 + [\frac{1}{2T}])^{1/2}$ *and* $\delta \in (0, 1]$ *such that*

$$\left(1 - \frac{\nu}{\mu - 2}\right) b\delta - M\delta^{\mu-1} = \max_{x\in[0,1]}\left(\left(1 - \frac{\nu}{\mu - 2}\right) bx - Mx^{\mu-1}\right). \tag{1.3}$$

Then, system (HS) possesses a nontrivial homoclinic solution.

Remark 1.1. Since $\nabla K(t, x) = \nabla[K(t, x) - K(t, 0)]$, so we can replace $K(t, x)$ by $K(t, x) - K(t, 0)$ if $K(t, 0) \ne 0$. Furthermore, we can treat $W(t, x)$ by the same method.

Remark 1.2. Conditions (H3) and (H4) are weaker than (V5) because condition (V5) implies that $\lim_{|x|\to\infty} \frac{W(t,x)}{|x|^2} = +\infty$.

In order to receive a homoclinic solution of (HS), similarly to paper [9], we consider a sequence of differential equations:

$$\ddot{u}(t) + \nabla V(t, u(t)) = f_k(t), \tag{HS$_k$}$$

where $f_k : R \to R^n$ is a $2kT$-periodic extension of restriction of f to the interval $[-kT, kT]$, $k \in N$. We will prove the existence of one homoclinic solution of (HS) as the limit of the $2kT$-periodic solutions of (HS$_k$) as in [9]. However, some technical details in this paper are different from [9].