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Finite time blow-up and global solutions for a class of semilinear parabolic equations at high energy level

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ABSTRACT

In this paper we study the initial boundary value problem of a class of semilinear parabolic equation. Our main tools are the comparison principle and variational methods. In this paper, we will find both finite time blow-up and global solutions at high energy level.

1. Introduction

In 1972, Tsutsumi [1] studied the problem of nonlinear parabolic equation

$$\frac{\partial u}{\partial t} = \sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p-2}\frac{\partial u}{\partial x_i}\right) + u^{1+\alpha}, \quad x \in \Omega,\ t > 0, \tag{1.1}$$

$$u(x, 0) = u_0(x), \quad x \in \Omega, \tag{1.2}$$

$$u(x, t) = 0, \quad x \in \partial\Omega,\ t \geq 0, \tag{1.3}$$

where $\Omega \subset \mathbb{R}^N$ is an open bounded domain. And $\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p-2}\frac{\partial u}{\partial x_i}\right)$ is called p-Laplace operator. For the case $p < 2+\alpha$, he obtained the existence of global weak solutions in the application of potential well. In 2004, Tao Zhong and Yao Zheng-an [2] changed the nonlinearity $u^{1+\alpha}$ into $|u|^{q-2}u$. And they proved that the global solutions with the initial data in some "*stable set*" converge strongly to zero in $W_0^{1,p}(\Omega)$. Then, using the potential well, Liu Yacheng and Zhao Junsheng [3] proved that if $0 \leq u_0(x) \in W_0^{1,p}(\Omega)$, $J(u_0) = d$, $I(u_0) > 0$ or $I(u_0) = 0$, $0 < J(u_0) \leq d$, then the corresponding problem admits a global solution $u(t) \in L^\infty(0, \infty; W_0^{1,p}(\Omega))$ with $u_t(t) \in L^2(0, \infty; L^2(\Omega))$ and $u(t) \in \overline{W}$ for $0 \leq t < \infty$ exists.

So far the situation $J(u_0) \leqslant d$ has been discussed. But the depth of potential well "d" for the problem (1.1)–(1.3) is usually very small so that such initial data u_0 do not satisfy the high energy case $J(u_0) > d$. In fact, many people have discussed the high energy level conditions for some kinds of partial differential equations. In 2009, Chen Shangjie and Tang Chunlei [4] studied the existence of infinitely many large energy solutions for the superlinear Schrödinger–Maxwell equations. In 2005, Gazzola and Weth [5] found finite time blow-up and global solutions for a kind of parabolic equations at high energy level.

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In this paper we will also use the way in [5] to consider the equation with high energy level

$$\frac{\partial u}{\partial t} = \sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p-2} \frac{\partial u}{\partial x_i}\right) + u^{1+\alpha}, \tag{1.4}$$

where p, α satisfy

$$p < 2 + \alpha \quad \text{if } N \le p; \qquad p < 2 + \alpha \le \frac{Np}{N-p} \quad \text{if } N > p. \tag{1.5}$$

And, we will prove that we can both find finite time blow-up and global solutions while $J(u_0) > d$.

2. Setup and notations

We denote the $L^q(\Omega)$ norm by $\|\cdot\|_q$ for $1 \le q \le \infty$ and by $\|\nabla \cdot\|_p$ the Dirichlet norm in $W_0^{1,p}(\Omega)$. And we define the cone of nonnegative functions

$$\mathbb{K} = \{u \in W_0^{1,p}(\Omega) \mid u \ge 0 \text{ a.e. in } \Omega\}.$$

For any $u \in W_0^{1,p}(\Omega)$, we denote its positive part by

$$u^+(x) := \max\{u(x), 0\},$$

and its negative part by

$$u^-(x) := \min\{u(x), 0\}.$$

We also define the energy functional and the Nehari functional

$$J(u) = \frac{1}{p}\|\nabla u\|_p^p - \frac{1}{2+\alpha}\|u\|_{2+\alpha}^{2+\alpha}, \qquad I(u) = \|\nabla u\|_p^p - \|u\|_{2+\alpha}^{2+\alpha}. \tag{2.1}$$

By the Nehari manifold, we define

$$\mathcal{N} = \{u \in W_0^{1,p}(\Omega) \setminus \{0\} | I(u) = 0\},$$

and the unbounded sets separated by $\mathcal{N}$

$$\mathcal{N}_+ = \{u \in W_0^{1,p}(\Omega) | I(u) > 0\} \quad \text{and} \quad \mathcal{N}_- = \{u \in W_0^{1,p}(\Omega) | I(u) < 0\}.$$

In the following, we also let $T^*(u_0)$ denote the maximal existence time of the solution with initial condition $u_0 \in W_0^{1,p}(\Omega)$. And we denote by $S(t)$ the nonlinear semigroup associated to (1.1). Therefore, we will write $S(t)u_0$ instead of $u = u(t)$ for $t < T^*(u_0)$.

In fact, when the equation is fixed, whether $u(x)$ global exists or blows up in finite time is just determined by the initial data $u_0(x)$. Following this consideration, let us introduce the sets

$$\mathcal{B} = \{u_0 \in W_0^{1,p}(\Omega) | \text{ the solution } u = u(t) \text{of (1.1)–(1.3) blows up in finite time}\},$$

$$\mathcal{G} = \{u_0 \in W_0^{1,p}(\Omega) | T^*(u_0) = \infty\},$$

$$\mathcal{G}_0 = \{u_0 \in \mathcal{G} | u(t) \to 0 \text{ in } W_0^{1,p}(\Omega) \text{ as } t \to \infty\}.$$

3. Comparison principle and a theorem about the stationary problem

In this section, we will also discuss the stationary problem

$$-\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p-2} \frac{\partial u}{\partial x_i}\right) = u^{1+\alpha} \quad \text{in } \Omega, \tag{3.1}$$

$$u = 0 \quad \text{on } \partial\Omega. \tag{3.2}$$

Lemma 3.1. *Let $u_0 \in W_0^{1,p}(\Omega)$ be such that $T^*(u_0) = \infty$. Then, we have the convergence of solution $S(t)u_0$ to the solution of problem (3.1)–(3.2).*

The lemma has been proved by Chill, Fiorenza [7] and Guesmia [6].

In this paper, the methods in [5] are employed to prove the main results but the comparison principle is not ready for us to complete the main proof. So we first give the following comparison principle Lemma 3.3 for initial data $u_0 \in W_0^{1,p}(\Omega)$. And to prove this principle, we will introduce Lemma 3.2 first.