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Family of equilibria in a population kinetics model and its collapse

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ABSTRACT

The system of nonlinear parabolic equations that models the dynamics of three populations is studied. The strong nonuniqueness in the form of continuous cosymmetric family of steady states is detected for some parameter values. Collapse of the family of equilibria under perturbation of boundary conditions is analyzed numerically.

1. Introduction

The existence of many practically identical steady states in some biological and economical models is a very interesting situation. Symmetry may lead to the strong nonuniqueness in the form of continuous family of steady states, another reason is cosymmetry [1,2]. A number of problems in mathematical physics are cosymmetric ones [3,4], e.g. Darcy convection of an incompressible fluid saturating a porous medium, some models of anti-ferromagnetism. Cosymmetry is an essentially nonlinear effect that gives the emergence of continuous families of steady states and nontrivial dynamics. In contrast to symmetry problems, equilibria belonging to the family have variative spectrum of stability. Each such state may be realized from individual initial conditions. Even the cosymmetry property was destroyed under some perturbations the dynamics is being very close to the vanished states [3–6]. So, it is very important to investigate cosymmetric system as some idealization that can provide a better understanding of a real system.

We study the dynamics of a system of nonlinear parabolic equations modelling the behavior of three populations which inhabit a one-dimensional domain [7]. The resulting system is a cosymmetric one for the Dirichlet boundary conditions and has a solution in the form of a continuous family of steady states. Under perturbation of boundary conditions the system loses linear cosymmetry and then the family of equilibria is destroyed. We analyze these effects by numerical simulation and detect different transformations of a cosymmetric family of equilibria to a number of isolated regimes.

The outline of the paper is as follows. In Section 2 we specify the population dynamics problem and discuss the properties of the underlying system of partial differential equations. In Section 3 we briefly describe the semi-discretization of the problem and solution method. The results of computation for different regimes and the analysis of regime transformations are presented in Section 4.

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2. Population kinetics model

Dynamics of three populations (species) sharing a common interval $\Omega = [0, a]$ is described by a system of nonlinear parabolic equations [7]

$$\dot{w} = Kw'' + Mw' + F(w', w) \equiv \Phi(w), \tag{1}$$

where $w = (w_1, w_2, w_3)^\top$ is a vector of density deviations (deviation from averaged value), the dot and prime mean differentiation with respect to time t and space coordinate $x \in \Omega$, respectively.

The right-hand side of (1) is composed of the diffusion term Kw'', the transportation term Mw', and the nonlinear interaction term $F(w, w')$. The diffusion is given by second order derivatives and the diagonal matrix of diffusive coefficients $K = \mathrm{diag}(k_1, k_2, k_3)$. The transport term Mw' is formed using the matrix with transport parameters (migration intensities) ν and λ

$$M = \begin{pmatrix} 0 & \nu & -\lambda \\ \nu & 0 & 0 \\ \lambda & 0 & 0 \end{pmatrix}, \tag{2}$$

We use below also the notation m_{ik} for the matrix elements, where $m_{21} = m_{12} = \nu$, $m_{31} = -m_{13} = \lambda$ and all other m_{ik} vanish. The nonlinear interaction term F is given by the bilinear form:

$$F(u, v) = \eta K \begin{pmatrix} -3u_1v_1 \\ u_1v_2 + 2u_2v_1 \\ u_1v_3 + 2u_3v_1 \end{pmatrix}. \tag{3}$$

The real parameter η is introduced to govern the nonlinear interaction.

The initial conditions of the problem are

$$w(x, 0) = w^0(x), \quad x \in [0, 1]. \tag{4}$$

We consider Eqs. (1)–(4) with the boundary conditions

$$w(0, t) = \xi, \qquad w(1, t) = \gamma, \tag{5}$$

where $\xi = (\xi_1, \xi_2, \xi_3)^\top$, $\gamma = (\gamma_1, \gamma_2, \gamma_3)^\top$. The case $\xi = \gamma = 0$ corresponds to the nullification of density deviation on the boundary. So, nonzero ξ or/and γ allow us to study perturbations of density.

We will consider the system (1)–(4) mainly with boundary conditions

$$w(0, t) = w(1, t) = \xi. \tag{6}$$

Obviously the problem (1)–(4), (5) has no stationary solution $w = \xi + x\chi$, $\chi = \gamma - \xi$ when $\gamma \neq \xi$. Substituting to (1) gives

$$0 = M\chi + F(\chi, \xi) + xF(\chi, \chi).$$

Consequently, $\gamma = \xi$.

The main parameters of the problem are the transport parameters λ and ν, the diffusivity coefficients k_j, the growth parameter η and concentrations on the boundary ξ_j and γ_j.

2.1. Cosymmetry property

In [8] the system (1)–(4) was studied under Dirichlet boundary conditions

$$w(0, t) = w(1, t) = 0. \tag{7}$$

Here we refer to the problem (1)–(4), (7) as an unperturbed problem (or cosymmetric system) with has the linear cosymmetry [7]

$$\Psi = BK^{-1}Mw, \quad B = \mathrm{diag}(1, -1, 1). \tag{8}$$

This means that

$$(\Phi, \Psi)_{L_2} = \int_\Omega (\Phi, \Psi)\mathrm{d}x = 0. \tag{9}$$

Moreover, the following identities are valid:

$$\int_\Omega (Kw'', \Psi)\mathrm{d}x = 0, \tag{10}$$

$$\int_\Omega (Mw', \Psi)\mathrm{d}x = 0, \tag{11}$$

$$\int_\Omega (F(w, w'), \Psi)\mathrm{d}x = 0. \tag{12}$$