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# Periodicity and blowup in a two-species cooperating model ☆

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ABSTRACT

In this paper, the cooperating two-species Lotka–Volterra model is discussed. The existence and asymptotic behavior of $T$-periodic solutions for the periodic reaction diffusion system under homogeneous Dirichlet boundary conditions are first investigated. The blowup properties of solutions for the same system are then given. It is shown that periodic solutions exist if the intra-specific competitions are strong whereas blowup solutions exist under certain conditions if the intra-specific competitions are weak. Numerical simulations and a brief discussion are also presented in the last section.



## 1. Introduction

This paper deals with the two-species Lotka–Volterra model under Dirichlet boundary conditions:

$$\begin{cases} u_t - d_1\Delta u = u[a_1 - b_1 u(x,t) + c_1 v(x,t)], & (x,t)\in\Omega\times(0,+\infty),\\ v_t - d_2\Delta v = v[a_2 + b_2 u(x,t) - c_2 v(x,t)], & (x,t)\in\Omega\times(0,+\infty),\\ u(x,t) = v(x,t) = 0, & x\in\partial\Omega\times(0,+\infty) \end{cases} \tag{1.1}$$

with the periodic conditions

$$u(x,0) = u(x,T), \qquad v(x,0) = v(x,T), \quad x\in\Omega \tag{1.2}$$

and also under the initial conditions

$$u(x,0) = \eta_1(x) \ge 0, \qquad v(x,0) = \eta_2(x) \ge 0, \quad x\in\Omega, \tag{1.3}$$

where $u$, $v$ represent the spatial density of the two cooperating species at time $t$ and $d_i \equiv d_i(x,t)$, $a_i \equiv a_i(x,t)$, $b_i \equiv b_i(x,t)$, $c_i \equiv c_i(x,t)$ $(i = 1, 2)$ are smooth positive $T$-periodic functions on $\Omega\times(0,+\infty)$. $\Omega$ is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$. Here we assume that the presence of one species encourages the growth of the preceding one and vice versa. The zero Dirichlet boundary condition corresponds to the assumption that crossing the boundary is lethal to the species and hence the species cannot exist on the boundary. We are interested in the behavior of the $T$-periodic solution as well as the asymptotic behavior of problem (1.1) (1.3) in relation to the maximal and minimal $T$-periodic solution of system (1.1) (1.2).

Many articles [1–4] have investigated the periodic solutions of parabolic boundary value problems and various methods have been proposed for the existence and qualitative properties of the solutions. Because of the periodicity of the birth and death rates, rates of diffusion, rates of interactions and environmental carrying capacities on seasonal scale, nonlinear periodic diffusion equations arise naturally in population models [5]. Hess [6] studied the existence and global stability

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of a $T$-periodic solution of periodic boundary value problem of the logistic model. Zhao [7] considered a general periodic parabolic system of competitor–competitor–mutualist model with spatial heterogeneity and subject to a general boundary condition, and studied the global asymptotic behavior of the system mainly by parabolic periodic eigenvalue theory, uniform persistence theory and monotone dynamical system theory. Pao [8] developed the method of construction of a pair of upper and lower solutions and its associated monotone iterations for a general class of strongly coupled elliptic systems. The monotone iterative scheme associated with this method leads to various computation algorithms for numerical solutions of the periodic boundary problem. The stability and attractivity analysis which are for quasimonotone nondecreasing and mixed quasimonotone reaction functions by the monotone iteration scheme were given in [9,10].

Now the question arises whether the solution of the reaction diffusion system with periodic parameters is periodic or not. Of course the answer is no and one possibility is that the solution blows up as time increases even though all parameters are periodic. Therefore we will also consider the blowup properties of solutions to the same system

$$\begin{cases} \dfrac{\partial u}{\partial t} - d_1(x,t)\Delta u = u[a_1(x,t) - b_1 u(x,t) + c_1 v(x,t)] & \text{in } \Omega_{T'}, \\ \dfrac{\partial v}{\partial t} - d_2(x,t)\Delta v = v[a_2(x,t) + b_2 u(x,t) - c_2 v(x,t)] & \text{in } \Omega_{T'}, \\ u(x,t) = v(x,t) = 0 & \text{on } \partial\Omega_{T'}, \\ u(x,0) = \eta_1(x), \qquad v(x,0) = \eta_2(x), & \text{in } \Omega_{T'}, \end{cases} \tag{1.4}$$

where all the parameters $d_i(x,t)$, $a_i(x,t)$, $b_i(x,t)$, $c_i(x,t)$ $(i = 1,2)$ are smooth positive $T$-periodic functions on $\Omega \times (0,+\infty)$. The domains $\Omega_{T'}$ and $\partial\Omega_{T'}$ are defined as, respectively, $\Omega \times (0,T')$ and $\partial\Omega \times (0,T')$, $T'$ is the maximal existence time of the solution. $\eta_1$, $\eta_2$ are smooth functions satisfying the compatibility condition $\eta_1(x) = 0$ and $\eta_2(x) = 0$ for $x \in \partial\Omega$. Here we say the solution $(u,v)$ blows up in a finite time $T^* > 0$ if

$$\lim_{t\to T^*} \max_{\overline{\Omega}} (|u(\cdot,t)| + |v(\cdot,t)|) = +\infty.$$

Recently, there is also much research work [11–15] to blowup of the solutions for the problems modelling the ecological models. For example, Pao [15] considered the following reaction diffusion system

$$\begin{cases} u_t - D_1\Delta u = u(a_1 - b_1 u + c_1 v) & \text{in } \Omega_T, \\ v_t - D_2\Delta v = v(a_2 + b_2 u - c_2 v) & \text{in } \Omega_T, \\ Bu = 0, \qquad Bv = 0 & \text{on } \partial\Omega_T, \\ u(x,0) = u_0(x), \qquad v(x,0) = v_0(x) & \text{in } \Omega \end{cases} \tag{1.5}$$

and proved that a unique solution $(u,v)$ of (1.5) exists and is uniformly bounded in $\overline{\Omega} \times [0,+\infty)$ when $b_1c_2 > b_2c_1$, while the solution $(u,v)$ may blow up in finite time when $b_1c_2 < b_2c_1$. Lou et al. [14] considered the problem (1.5) with homogeneous Neumann boundary conditions and gave a sufficient condition on the initial data for the solution to blow up in a finite time. Some sufficient conditions on global existence and finite time blowup of the solutions are described via all the six nonlinear exponents appearing in the six nonlinear terms.

Based on the above results, we are also interested in studying the blowup properties of the solutions. If

$$b_1^M c_2^M < b_2^L c_1^L, \tag{1.6}$$

then the solution of (1.4) with any nontrivial nonnegative data will blow up provided that $a_i^L > d_i^L\lambda$ or the initial data is large enough, where we denote $f^L = \inf_{t\in(0,T)} f(t)$ and $f^M = \sup_{t\in(0,T)} f(t)$ for any positive function $f(t)$.

This paper is arranged as follows. The next section deals with the existence of $T$-periodic solutions of system (1.1) (1.2). The stability and attractivity of the maximal and minimal $T$-periodic solutions and a global attractor of the system are established under some assumptions. Section 3 is devoted to the blowup of the solution. We first give several lemmas and then obtain sufficient conditions for the solution to blow up in finite time. Numerical simulations are given in Section 4 and a brief discussion is also given in the last section.

## 2. Existence of periodic solution

In this section, we study the periodic solutions of the problem (1.1) (1.2). First we give the definition of ordered upper and lower solutions of (1.1) (1.2) and then a pair of ordered upper and lower solutions is constructed. At last the properties of the periodic solutions are established.

We first consider the periodic eigenvalue problem

$$\begin{cases} \dfrac{\partial \phi}{\partial t} - L\phi - a\phi = \lambda\phi, & (x,t) \in \Omega \times (0,\infty), \\ B\phi = 0, & (x,t) \in \partial\Omega \times (0,\infty), \\ \phi(x,0) = \phi(x,T), & x \in \Omega, \end{cases} \tag{2.1}$$