Nonlinear Analysis: Real World Applications 12 (2011) 545–561

Contents lists available at ScienceDirect

Nonlinear Analysis: Real World Applications

journal homepage: www.elsevier.com/locate/nonrwa

A full-Newton step $O(n)$ infeasible-interior-point algorithm for linear complementarity problems

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ARTICLE INFO

Article history:
Received 6 January 2010
Accepted 28 June 2010

Keywords:
Linear complementarity problems
Feasible interior-point method
Infeasible interior-point method
Polynomial complexity

ABSTRACT

This paper consists of two parts. In the first part we present a new primal-dual feasible interior-point algorithm for solving monotone linear complementarity problems (LCP). Since the algorithm uses only full-Newton steps, it has the advantage that no line-searches are needed. It is proven that the number of iterations of the algorithm is $O\left(\sqrt{n}\log\frac{n}{\varepsilon}\right)$, which coincides with the well-known best iteration bound for LCP. In the second part, we generalize an infeasible interior-point method for linear optimization introduced by Roos (2006) [15] to LCP. Two types of full-Newton steps are used, feasibility steps and (ordinary) centering steps, respectively. The algorithm starts from strictly feasible iterates of a perturbed problem, on its central path, and feasibility steps find strictly feasible iterates for the next perturbed problem. By using centering steps for the new perturbed problem, we obtain strictly feasible iterates close enough to the central path of the new perturbed problem. The starting point depends on two positive numbers ρ_p and ρ_d. The algorithm terminates either by finding an ε-solution or detecting that the LCP problem has no optimal solution with vanishing duality gap satisfying a condition in terms of ρ_p and ρ_d. The iteration bound coincides with the currently best iteration bound for linear complementarity problems.

1. Introduction

The monotone linear complementarity problem (LCP) is to find a vector pair $(x, s) \in \mathbf{R}^{2n}$ that satisfies the following conditions

$$s = Mx + q, \quad (x, s) \geq 0, \; x^T s = 0, \tag{P}$$

where $q \in \mathbf{R}^n$ and M is an $n \times n$ matrix supposed positive semidefinite. Interior-point methods (IPMs) for solving Linear Optimization (LO) problems were initiated by Karmarkar [1]. They not only have polynomial complexity, but are also highly efficient in practice. One may distinguish between feasible IPMs and infeasible IPMs (IIPMs). Feasible IPMs start with a strictly feasible interior-point and maintain feasibility during the solution process. Feasible IPMs require that the starting points satisfy exactly the equality constraints and are strictly positive, i.e., they lie in the interior of a region defined by inequality constraints. All subsequent points generated by the feasible IPMs will have the same properties.

Extending methods for LO to LCP has been successful in many cases. See, e.g., [2–4]. Recently, Peng et al. [5,6] designed primal–dual feasible IPMs by using self-regular functions for LO and also extended the approach to LCP. The complexity bounds obtained by these authors are $O\left(\sqrt{n}\right)\log\frac{n}{\varepsilon}$ and $O\left(\sqrt{n}\log n\right)\log\frac{n}{\varepsilon}$, for small-update methods and large-update methods, respectively, which are currently the best known iteration bounds.

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doi:10.1016/j.nonrwa.2010.06.039

IIPMs start with an arbitrary positive point and feasibility is reached as optimality is approached. The choice of the starting point in IIPMs is crucial for the performance. Lustig [7] and Tanabe [8] were the first to present IIPMs for LO. Kojima et al. [9] were the first that proved the global convergence of a primal–dual IIPM for LO. Zhang [10] was the first who presented a primal–dual IIPM with polynomial iteration complexity $O\left(n^2 \log \frac{1}{\varepsilon}\right)$ for LO. Mizuno [11] and Potra [12,13] also introduced a primal–dual IIPM for LO with polynomial iteration complexity $O\left(n \log \frac{1}{\varepsilon}\right)$. Potra [3] analyzed a generation to LCP of the Mizuno–Todd–Ye predictor corrector method [14] for infeasible starting points and he proved that the complexity of his algorithm is

$$O\left(n \log \frac{\max\left\{\left\|r^0\right\|, \left(x^0\right)^T s^0\right\}}{\varepsilon}\right), \tag{1}$$

which is currently the best known iteration bound of IIPMs for LCP. Here r^0 denotes the initial value of the residual vector as defined in Section 4.1. It is assumed in this result that there exists an optimal solution (x^*, s^*) for (P) as defined in Section 5.2 such that

$$n^{-1}\left\|x^*\right\|_1 \le \rho_p, \qquad \max\left\{n^{-1}\left\|s^*\right\|_1, \rho_p \left\|Me\right\|_\infty, \left\|q\right\|_\infty\right\} \le \rho_d,$$

and the starting point is

$$x^0 = \rho_p e, \qquad s^0 = \rho_d e,$$

where e is denoted the all-one vector.

In this paper we first present a new feasible primal–dual IPM with full-Newton steps for LCP problems. We prove that the complexity of our algorithm is $O\left(\sqrt{n} \log \frac{n}{\varepsilon}\right)$ which coincides with the best known iteration bound for feasible IPMs. Then we extend the algorithm presented by Roos [15] for LCP problems. We prove that the complexity of our algorithm coincides with the best known iteration bound for IIPMs as given in (1).

The paper is organized as follows: In Section 2, we study some basic concepts for feasible IPMs for solving the LCP problems, such as central path, full-Newton step, etc. We also present a primal–dual feasible IPM for LCP. In Section 3, we find a sufficient condition for feasibility of full-Newton steps and also quadratic convergence to target points on the central path is proved. In Section 3.3, the complexity bound for our feasible algorithm is derived. Section 4 is used to describe our infeasible algorithm in more detail. One characteristic of the algorithm is that it uses an intermediate problem. The intermediate problem is a suitable perturbation of the given problem (P) so that at any stage the iterate is strictly feasible for the current perturbed problem; the size of the perturbation decreases at the same speed as the barrier parameter μ. When μ changes to a smaller value, the perturbed problem corresponding to μ changes, and hence also the current central path. The algorithm keeps the iterate close to μ-center on the central path of the current perturbed problem. To get the iterate feasible for the new problem and close to its central path, we use a so-called 'feasibility step'. The largest, and hardest, part of the analysis, which is presented in 5, concerns this step. It turns out that to keep control over this step, before taking the step, the iterates need to be very well centered. Some concluding remarks can be found in Section 6.

The notations used throughout the paper are rather standard: capital letters denote matrices, lower case letters denote vectors, script capital letters denote sets, and Greek letters denote scalars. All vectors are considered to be column vectors. The components of a vector $u \in \mathbf{R}^n$ will be denoted by $u_i, i = 1, \ldots, n$. The relation $u > 0$ is equivalent to $u_i > 0, i = 1, \ldots, n$, while $u \ge 0$ means $u_i \ge 0, i = 1, \ldots, n$. We denote $\mathbf{R}^n_+ = \{u \in \mathbf{R}^n : u \ge 0\}$, $\mathbf{R}^n_{++} = \{u \in \mathbf{R}^n : u > 0\}$. For any vector $x \in \mathbf{R}^n$, $x_{\min} = \min(x_1; x_2; \ldots; x_n)$ and $x_{\max} = \max(x_1; x_2; \ldots; x_n)$. If $u \in \mathbf{R}^n$ then $U := \operatorname{diag}(u)$ denotes the diagonal matrix having the components of u as diagonal entries. If $x, s \in \mathbf{R}^n$, then xs denotes the componentwise (Hadamard) product of the vectors x and s. Furthermore, e denotes the all-one vector of length n. The 2-norm and the infinity norm for vectors are denoted by $\|\cdot\|$ and $\|\cdot\|_\infty$, respectively. The Frobenius matrix norm is given by $\|U\|^2 := \sum_{i=1}^m \sum_{j=1}^n U_{ij}^2 = \mathbf{Tr}\left(U^T U\right)$. We denote the feasible set of the problem (P) by

$$\mathcal{F} := \left\{(x, s) \in \mathbf{R}^{2n}_+ : s = Mx + q\right\}$$

and its solution set by

$$\mathcal{F}^* := \left\{\left(x^*, s^*\right) \in \mathcal{F} : \left(x^*\right)^T s^* = 0\right\}.$$

Throughout this paper it will be assumed that $\mathcal{F}^*$ is not empty, i.e., (P) has at least one solution.

2. Feasible full-Newton step IPMs

To solve LCP one needs to find a solution of the following system of equations

$$\begin{array}{ll} s = Mx + q, & x \ge 0, \\ xs = 0, & s \ge 0. \end{array} \tag{2}$$

In these so-called optimality conditions the first constraint represents feasibility, whereas the last equation is the so-called complementary condition. The nonnegativity constraints in the feasibility condition make the problem already nontrivial: Only iterative methods can find a solution of a linear system involving inequality constraints. The complementary condition is nonlinear, which makes it extra hard to solve this system.