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Homoclinic solutions for a class of second-order p-Laplacian differential systems with delay[☆]

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ABSTRACT

By means of an extension of Mawhin's continuation theorem and some analysis methods, the existence of a set with $2kT$-periodic solutions for a class of second-order p-Laplacian systems with delay is studied, and then a homoclinic solution is obtained as a limit of a certain subsequence of the above set.

1. Introduction

In this paper, we investigate the existence of homoclinic solutions for a class of second-order p-Laplacian differential systems with delay as follows:

$$\frac{\mathrm{d}}{\mathrm{d}t}[\varphi_p(u'(t))] + \frac{\mathrm{d}}{\mathrm{d}t}\nabla F(u(t)) + \nabla G(u(t)) + \nabla H(u(t-\gamma(t))) = e(t), \tag{1.1}$$

where $p \in (1, +\infty)$, $\varphi_p : R^n \to R^n \varphi_p(u) = (|u_1|^{p-2}u_1, |u_2|^{p-2}u_2, \ldots, |u_n|^{p-2}u_n)$ for $u \neq O = (0, 0, \ldots, 0)$ and $\varphi_p(O) = (0, 0, \ldots, 0)$, $F \in C^2(R^n, R)$, $G, H \in C^1(R^n, R)$, $e \in C(R, R^n)$ and $\gamma(t)$ is a continuous T-periodic function with $\gamma(t) \geq 0$, $T > 0$ is a given constant.

As is well known, a solution $u(t)$ of Eq. (1.1) is named homoclinic (to O) if $u(t) \to O$ and $u'(t) \to O$ as $|t| \to +\infty$. In addition, if $u \not\equiv O$, then u is called a nontrivial homoclinic solution.

The existence of homoclinic solutions for some second-order ordinary differential equations has been extensively studied by using critical point theory; see [1–4] and the references therein. In [5], by using the methods of bifurcation theory, the authors investigated the existence of homoclinic solutions to some retarded functional differential equations with parameters. However, as far as we know, few papers have discussed the existence of homoclinic solutions to retarded functional differential equations without parameters. For Eq. (1.1), even if $\gamma(t) \equiv 0$, since the equation contains the term $\frac{\mathrm{d}}{\mathrm{d}t}\nabla F(u(t))$, the method of critical point theory in [1–4] cannot be applied directly. This is due to the fact that the differential system is not the Euler–Lagrange equation associated with some functional $I : W^{1,p}_{2kT} \to R$.

Like in the work of Rabinowitz in [6], Marek Lzydorek and Joanna Janczewska in [7] and X.H. Tan and Li Xiao in [8], the existence of a homoclinic solution for the equation is obtained as a limit of a certain sequence of $2kT$-periodic solutions for

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the following equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}[\varphi_p(u'(t))] + \frac{\mathrm{d}}{\mathrm{d}t}\nabla F(u(t)) + \nabla G(u(t)) + \nabla H(u(t-\gamma(t))) = e_k(t), \tag{1.2}$$

where $k \in \mathbf{N}$, $e_k : R \to R^n$ is a $2kT$-periodic function such that

$$e_k(t) = \begin{cases} e(t) & t \in [-kT, kT-\varepsilon_0), \\ e(kT-\varepsilon_0) + \dfrac{e(-kT) - e(kT-\varepsilon_0)}{\varepsilon_0}(t - kT + \varepsilon_0) & t \in [kT-\varepsilon_0, kT], \end{cases} \tag{1.3}$$

$\varepsilon_0 \in (0, T)$ is a constant independent of k. However, in our approach, the existence of $2kT$-periodic solutions to Eq. (1.2) is obtained by using an extension of Mawhin's continuation theorem [9], not by using the methods of critical point theory, which is quite different from the approach of [1–4,6–8]. Furthermore, the main result in the present paper is related to the value of the delay $\gamma(t)$, and also the methods for getting *a priori bounds* of periodic solutions for Eq. (1.2) are essentially different from the corresponding ones of [10–14].

2. Preliminaries

Throughout this paper, $|\cdot|$ will denote the absolute value and the Euclidean norm on R^n. For each $k \in \mathbf{N}$, let $C_{2kT} = \{x | x \in C(R, R^n), x(t+2kT) \equiv x(t)\}$, $C^1_{2kT} = \{x | x \in C^1(R, R^n), x(t+2kT) \equiv x(t)\}$ and $|x|_0 = \max_{t\in[0,2kT]} |x(t)|$. If the norms of C_{2kT} and C^1_{2kT} are defined by $\|\cdot\|_{C_k} = |\cdot|_0$ and $\|x\|_{C^1_{2kT}} = \max\{|x|_0, |x'|_0\}$, respectively, then C_{2kT} and C^1_{2kT} are all Banach spaces. Furthermore, for $\phi \in C_{2kT}$, $\|\phi\|_r = \left(\int_{-kT}^{kT} |\phi(t)|^r \mathrm{d}t\right)^{1/r}$, where $r \in (1, +\infty)$.

Lemma 2.1 (*[13]*)**.** *Let $s \in C(R, R)$ with $s(t+\omega) \equiv s(t)$ and $s(t) \in [0, \omega]$, $\forall t \in R$. Suppose $p \in (1, +\infty)$, $\alpha = \max_{t\in[0,\omega]} s(t)$ and $u \in C^1(R, R)$ with $u(t+\omega) \equiv u(t)$. Then*

$$\int_0^\omega |u(t) - u(t - s(t))|^p \mathrm{d}t \le \alpha^p \int_0^\omega |u'(t)|^p \mathrm{d}t.$$

Lemma 2.2. *If $q : R \to R^n$ is continuously differentiable on R, $a > 0$ and $p > 1$ are constants, then for every $t \in R$ the following inequality holds:*

$$|q(t)| \le (2a)^{-\frac{1}{p}} \left(\int_{t-a}^{t+a} |q(s)|^p \mathrm{d}s\right)^{1/p} + a(2a)^{-1/p} \left(\int_{t-a}^{t+a} |q'(s)|^p \mathrm{d}s\right)^{1/p}.$$

This lemma is a special case of Lemma 2.2 in [8].

In order to study the existence of $2kT$-periodic solutions for Eq. (1.2)*, for each $k \in \mathbf{N}$, from* (1.3) *we observe that $e_k \in C_{2kT}$. Let $X_k = C^1_{2kT}$.*

Lemma 2.3 (*[9]*)**.** *Assume that Ω is an open bounded set in X_k such that the following conditions are satisfied:*

[C$_1$] *For each $\lambda \in (0, 1)$, the equation*

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi_p[x'(t)] + \lambda\frac{\mathrm{d}}{\mathrm{d}t}\nabla F(u(t)) + \lambda\nabla G(u(t)) + \lambda\nabla H(u(t-\gamma(t))) = \lambda e_k(t)$$

has no solution on $\partial\Omega$.

[C$_2$] *The equation*

$$\Delta(a) := \frac{1}{2kT}\int_0^{2kT} [\nabla G(a) + \nabla H(a) - e_k(t)]\mathrm{d}t = 0$$

has no solution on $\partial\Omega \cap R^n$.

[C$_3$] *The Brouwer degree*

$$d_B\{\Delta, \Omega \cap R^n, 0\} \neq 0.$$

Then Eq. (1.2) *has a $2kT$-periodic solution in $\overline{\Omega}$.*

Lemma 2.4. *If $x \in (0, +\infty)$ satisfies the following inequality:*

$$x^s \le \alpha x^q + \beta x^r,$$