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Analytic solutions for generalized forms of the nonlinear heat conduction equation

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ABSTRACT

An analytic study of the two generalized forms of the nonlinear heat conduction equation is presented in this paper. The (G'/G)-expansion method and the Exp-function method are employed to derive exact solutions of these equations. The solutions gained from each of the proposed methods have been verified with those obtained by the tanh method. More importantly, other new and more general solutions are, likewise, found for these equations which are of great practical importance in physical and engineering problems.

1. Introduction

"*The most incomprehensible thing about the world is that it is at all comprehensible*" (Albert Einstein), but the question is how do we fully understand incomprehensible things? Nonlinear science provides some clues in this regard [1].

The world around us is inherently nonlinear. For instance, nonlinear evolution equations (NLEEs) are widely used as models to describe complex physical phenomena in various fields of sciences, especially in fluid mechanics, solid-state physics, plasma physics, plasma waves, and biology. One of the basic physical problems for these models is to obtain their travelling wave solutions. In particular, various methods have been utilized to explore different kinds of solutions of physical models described by nonlinear partial differential equations (PDEs). For instance, in the numerical methods, stability and convergence should be considered, so as to avoid divergent or inappropriate results. However, in recent years, a variety of effective analytical and semi-analytical methods have been developed to be used for solving nonlinear PDEs, such as the variational iteration method (VIM) [2–4], the homotopy perturbation method (HPM) [5–7], the parameter-expansion method [8], the optimal homotopy asymptotic method (OHAM) [9], the homotopy analysis method (HAM) [10], the tanh method [11–13], the homogeneous balance method [14], the inverse scattering method [15], and others. Likewise, He and Wu [16] proposed a straightforward and concise method called the Exp-function method to obtain the exact solutions of NLEEs. The method, with the aid of Maple or Matlab, has been successfully applied to many kinds of NLEE [17–23]. Lately, the (G'/G)-expansion method, first introduced by Wang et al. [24], has become widely used to search for various exact solutions of NLEEs [25–30]. The results reveal that the two recent methods are powerful techniques for solving nonlinear partial differential equations (NPDEs) in terms of accuracy and efficiency. This is important, since systems of NPDEs have many applications in engineering.

The generalized forms of the nonlinear heat conduction equation can be given as

$$u_t - a\,(u^n)_{xx} - u + u^n = 0, \quad a > 0,\ n > 1 \tag{1.1}$$

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and, in (2 + 1)-dimensional space,

$$u_t - a\,(u^n)_{xx} - a\,(u^n)_{yy} - u + u^n = 0. \tag{1.2}$$

Many authors have studied some types of solutions of these equations. Wazwaz [12] used the tanh method to find solitary solutions of these equations and a standard form of the nonlinear heat conduction equation (when $n = 3$ in Eq. (1.1)). Also, Fan [13] applied the solutions of the Riccati equation in the tanh method to obtain the travelling wave solution when $n = 2$ in Eq. (1.1). More recently, Kabir et al. [20] implemented the Exp-function method to find exact solutions of Eq. (1.1), and obtained more general solutions in comparison with Wazwaz's results.

Considering all the indispensably significant issues mentioned above, the objective of this paper is to investigate the travelling wave solutions of Eqs. (1.1) and (1.2) systematically, by applying the (G'/G)-expansion and the Exp-function methods. Some previously known solutions are recovered as well, and, simultaneously, some new ones are also proposed.

2. Description of the two methods

2.1. The (G'/G)-expansion method

Suppose that a nonlinear PDE, say in two independent variables x and t, is given by

$$P(u, u_t, u_x, u_{xx}, u_{tt}, u_{tx}, \ldots) = 0, \tag{2.1}$$

or in three independent variables x, y, and t, is given by

$$P(u, u_t, u_x, u_y, u_{xx}, u_{yy}, u_{tt}, u_{tx}, u_{ty}, \ldots) = 0, \tag{2.2}$$

where P is a polynomial in its arguments, which include nonlinear terms and the highest-order derivatives.

Introducing a complex variable η defined as

$$u(x, t) = U(\eta), \quad \eta = k(x - ct) \tag{2.3}$$

or

$$u(x, y, t) = U\,(\eta), \quad \eta = k(x + y - ct), \tag{2.4}$$

Eq. (2.1) and (2.2) reduce to the ordinary differential equations (ODEs)

$$P(U, -kcU', k\,U', k^2U'', k^2c^2U'', -k^2cU'', \ldots) = 0, \tag{2.5}$$

and

$$P(U, -kcU', k\,U', k\,U', k^2U'', k^2U'', k^2c^2U'', -k^2cU'', -k^2cU'', \ldots) = 0, \tag{2.6}$$

respectively, where k and c are constants to be determined later. According to the (G'/G)-expansion method, it is assumed that the travelling wave solution of Eq. (2.5) or (2.6) can be expressed by a polynomial in $(\frac{G'}{G})$ as follows:

$$U(\eta) = \sum_{i=1}^{m} \alpha_i \left(\frac{G'}{G}\right)^i + \alpha_0, \qquad \alpha_m \neq 0, \tag{2.7}$$

where α_0 and α_i, for $i = 1, 2, \ldots, m$, are constants to be determined later, and $G(\eta)$ satisfies a second-order linear ordinary differential equation (LODE):

$$\frac{\mathrm{d}^2G(\eta)}{\mathrm{d}\eta^2} + \lambda\frac{\mathrm{d}G(\eta)}{\mathrm{d}\eta} + \mu G(\eta) = 0, \tag{2.8}$$

where λ and μ are arbitrary constants. Using the general solutions of Eq. (2.8), we have

$$\frac{G'(\eta)}{G(\eta)} = \begin{cases} \dfrac{\sqrt{\lambda^2 - 4\mu}}{2}\left(\dfrac{C_1 \sin h\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\eta\right) + C_2 \cos h\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\eta\right)}{C_1 \cos h\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\eta\right) + C_2 \sin h\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\eta\right)}\right) - \dfrac{\lambda}{2}, & \lambda^2 - 4\mu > 0, \\[2ex] \dfrac{\sqrt{4\mu - \lambda^2}}{2}\left(\dfrac{-C_1 \sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\eta\right) + C_2 \cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\eta\right)}{C_1 \cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\eta\right) + C_2 \sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\eta\right)}\right) - \dfrac{\lambda}{2}, & \lambda^2 - 4\mu < 0, \end{cases} \tag{2.9}$$