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The existence of homoclinic solutions for second-order Hamiltonian systems with periodic potentials

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ABSTRACT

In this paper, we study the existence of homoclinic solutions for the second-order Hamiltonian system $\ddot{u} - L(t)u + W_u(t, u) = 0$, where $L(t)$ and $W(t, u)$ are supposed to be periodic in t. Under certain assumptions on L and W, we obtain two new existence results by using the variant mountain pass theorem and generalized linking theorem.

1. Introduction and main results

In this paper, we consider the existence of homoclinic solutions for the following second-order Hamiltonian system:

$$\ddot{u} - L(t)u + W_u(t, u) = 0, \quad \forall\, t \in \mathbb{R}, \tag{1.1}$$

where $u = (u_1, \ldots, u_N) \in \mathbb{R}^N$, $W \in C^1(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$, and $L \in C(\mathbb{R}, \mathbb{R}^{N^2})$ is a symmetric T-periodic matrix-valued function. Here, as usual, we say that a solution u of system (1.1) is homoclinic (to 0) if $u \in C^2(\mathbb{R}, \mathbb{R}^N)$, $u(t) \not\equiv 0$, $u(t) \to 0$ and $\dot{u}(t) \to 0$ as $|t| \to \infty$.

The existence and multiplicity of homoclinic solutions for system (1.1) with or without the periodic property of $L(t)$ have been extensively investigated by the variational methods during the last two decades; for example, see [1–12] and the references therein. The case when $L(t)$ and $W(t, u)$ are either periodic in t or independent of t has been considered, for example, in [2–7,13]. In [2], the authors proved the existence of infinitely many homoclinic solutions without assuming that $W(t, u)$ is even in u. In [3,4], the authors proved the existence of a homoclinic solution as the limit of subharmonic solutions. In [5], the authors obtained the homoclinic solutions in the autonomous case by elementary minimization arguments. In [6], the authors developed a generalized linking theorem to deal with the case when zero lies in a spectral gap of the operator $-\ddot{u} + Lu$, and obtained the existence of homoclinic solutions for system (1.1). For the case when $L(t)$ and $W(t, u)$ are not necessarily periodic in t, we refer to [8–11,14,12] among many other papers.

It should be pointed out that, when $L(t)$ and $W(t, u)$ are periodic in t, the following global Ambrosetti–Rabinowitz (AR) condition or its slight generalization ((H5) in [13]) is supposed in all of the works (except [7]) mentioned above:

$$\exists\, \theta > 2 \text{ such that } 0 < \theta W(t, u) \le \langle W_u(t, u), u\rangle, \quad \forall\, t \in \mathbb{R} \text{ and } u \in \mathbb{R}^N \setminus \{0\}, \tag{1.2}$$

where $\langle \cdot, \cdot \rangle$ denotes the standard inner product in $\mathbb{R}^N$. The AR condition restricts the growth of $W(t, u)$ at both zero and infinity (e.g., see (3.2) and (3.3) in [8]), and it excludes some superlinear cases of $W_u(t, u)$ (Remark 1.2). Generally, the

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arguments in the above-mentioned works rely heavily on the condition. Motivated by [15], there has been some work to replace the condition by some less restrictive conditions [7]. Under a weaker condition, (W_3) below, the authors of [7], assuming that $s \mapsto s^{-1}\langle W_u(t, su), u\rangle$ is increasing for $s > 0$ (for all t and $u \neq 0$), proved the existence of homoclinic solutions by using the Nehari manifold. The strict monotone condition is also used in [5,16] with (without) the AR condition in other different situations.

When $L(t)$ is not periodic, problem (1.1) is different. One of the main methods to deal with it is to impose some coercive conditions on the eigenvalues of $L(t)$ and obtain the required compact imbedding results. Let us recall some facts. When $L(t)$ is positive definite for all t, by assuming the coercive condition $\inf_{|x|=1} L(t)x \cdot x \to \infty$ as $\|x\| \to \infty$, Rabinowitz [5] proved the existence of homoclinic solutions of (1.1). The compact imbedding theorem was proved in [9] under the same conditions. Further similar results were proved in [8] by establishing a different variational framework when $L(t)$ is not positive definite for all t. For some further results under the framework, see [10,11,17,14,12], where [17] does not assume the lower boundedness of $L(t)$.

In this paper, we only consider the case when $L(t)$ is periodic. Under a weaker monotone condition on W, we investigate the existence of homoclinic solutions for system (1.1) when $L(t)$ is positive definite, without assuming the AR condition (Theorem 1.1). We also consider the case when $L(t)$ is not positive definite and zero lies in a spectral gap of $-\ddot{u} + Lu$, and obtain the existence of ground-state homoclinic solutions (i.e., non-trivial solutions with least possible energy) of system (1.1) by using a generalized linking theorem (Theorem 1.2). Theorem 1.1 can be compared to Theorem 3.23 in [7], where in order to let their Nehari manifold method work they use a strict monotone condition (cf. (W_4)), while Theorem 1.2 can be compared to Theorem 5.1 in [6], where they use the AR condition and their homoclinic solution does not need to be a ground state.

For the statement of the first result, we make the following hypotheses.

(L_1) $L(t)$ is T-periodic in t and positive definite for all $t \in [0, T]$.

(W_1) $W(t, u)$ is T-periodic in t, $W(t, 0) \equiv 0$ and $W(t, u) \geq 0$ for $(t, u) \in \mathbb{R} \times \mathbb{R}^N$.

(W_2) $\lim_{|u|\to 0} \frac{W_u(t,u)}{|u|} = 0$ uniformly for $t \in \mathbb{R}$.

(W_3) $\lim_{|u|\to\infty} \frac{W(t,u)}{|u|^2} = \infty$ uniformly for $t \in \mathbb{R}$.

(W_4) $s^{-1}\langle W_u(t, su), u\rangle$ is a non-decreasing function of $s \in (0, 1]$, $\forall\,(t, u) \in \mathbb{R} \times \mathbb{R}^N$.

Theorem 1.1. *Under assumptions* (L_1), (W_1)–(W_4), *system* (1.1) *has at least one homoclinic solution.*

Next, we consider the case when $L(t)$ is not necessarily positive definite for all $t \in \mathbb{R}$ and make the following assumptions.

(L_2) $L(t)$ is T-periodic in t and 0 lies in a spectral gap of the operator $-\ddot{u} + Lu$.

(W_5) There exist $\mu > 2$, $c > 0$ such that

$$|W_u(t, u)| \leq c(|u|^{\mu-1} + 1), \quad \forall\,(t, u) \in \mathbb{R} \times \mathbb{R}^N.$$

(W_6) There exist constants $(\mu - 1) \leq \tau_1, \tau_2 \leq 2(\mu - 1)$, $a_1, a_2 > 0$ such that

$$\frac{1}{2}\langle W_u(t, u), u\rangle - W(t, u) \geq a_1|u|^{\tau_1}, \quad \forall\, t \in \mathbb{R},\ |u| \geq 1,$$

$$\frac{1}{2}\langle W_u(t, u), u\rangle - W(t, u) \geq a_2|u|^{\tau_2}, \quad \forall\, t \in \mathbb{R},\ |u| < 1.$$

Theorem 1.2. *Under assumptions* (L_2), (W_1)–(W_3), *and* (W_5)–(W_6), *system* (1.1) *has at least one ground-state homoclinic solution.*

Remark 1.1. Conditions (W_2) and (W_3) imply that $W(t, u)$ is superquadratic both at the origin and at infinity. A similar condition to (W_4) but with strict monotonicity is introduced for problem (1.1) in [5]. Condition (W_5) is a variant of the so-called non-quadratic condition introduced in [18]. For a comment on condition (L_2), see [1, p. 190].

Remark 1.2. There are functions $W(t, u)$ satisfying the conditions of Theorem 1.1, but not satisfying the AR condition; for example, $W_u(t, u) = W_u(u) = u \ln(|u| + 1)$ or $u \ln(|u|^2 + 1)$, etc. If we modify the definitions of the above functions near zero suitably, we can easily obtain functions satisfying the conditions of Theorem 1.1 but not satisfying the strict monotone condition required by the Nehari manifold method [7]. For example, let $W_u(u) = 0$ as $u \in [-1, +1]$, $W_u(u) = u \ln(|u| + 1)$ as $|u| \geq 2$, and connect 0 to $u \ln(|u| + 1)$ in $[-2, -1] \cup [1, 2]$ monotonically and smoothly.

Compared to the AR condition (1.2) or the condition $(F_2)\mu$ in [19], one advantage of condition (W_6) is that it allows the function $W(t, u)$ to have different growth at zero and infinity. Let

$$W(u) = \begin{cases} u^2 \ln(u^2 + 1), & |u| \geq 1, \\ (\ln 2)|u|^{\left(2 + \frac{1}{\ln 2}\right)}, & |u| < 1. \end{cases}$$

Choose $2 + 1/(2 \ln 2) < \mu < 3$, $\tau_1 = 2$ and $\tau_2 = 2 + (1/\ln 2)$. Then $\mu - 1 < \tau_1 < \tau_2 < 2(\mu - 1)$. We can verify that W satisfies the conditions of Theorem 1.2, but does not satisfy (1.2). Hence, Theorem 5.1 in [6] is not applicable.