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# An affirmative answer to the extended Gopalsamy and Liu's conjecture on the global asymptotic stability in a population model

Huaixing Li *, Rong Yuan

*School of Mathematical Sciences, Beijing Normal University, Beijing 100875, People's Republic of China*

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ABSTRACT

We study the global asymptotic stability of the positive equilibrium in a population model with a piecewise constant argument. Gopalsamy and Liu conjectured that the positive equilibrium $N^* = \frac{1}{a+b}$ is globally asymptotically stable if and only if the following inequality holds,

$$r \le \hat{\hat{r}}(\alpha) \equiv \frac{1+\alpha}{\alpha} \ln \frac{1+\alpha}{1-\alpha}$$

which has been solved by Muroya and Kato (2005) [2], Li and Yuan (2008) [1] for $\alpha := \frac{a}{b} \in [0, 1)$. But, for $\alpha \in (-1, 0)$, is the above inequality the necessary and sufficient condition for the global asymptotic stability of the positive equilibrium $\frac{1}{a+b}$? In this paper, we will give an affirmative answer to the extended Gopalsamy and Liu's conjecture.


## 1. Introduction

We consider the following differential equations with a piecewise constant argument:

$$\begin{cases} \dfrac{\mathrm{d}N(t)}{\mathrm{d}t} = rN(t)\{1 - aN(t) - bN([t])\}, \quad t > 0, \\ N(0) = N_0 > 0, \end{cases} \tag{1.1}$$

where $r, b > 0$ and $[t]$ denotes the maximal integer less than or equal to $t$.

Clearly, $N^* = \frac{1}{a+b}$ is the unique positive equilibrium of Eq. (1.1) if $a + b > 0$. For reader's convenience, we use the same notations as in [1,2]:

$$\alpha = \frac{a}{b}, \qquad t^* = \frac{\alpha}{1+\alpha}, \qquad t^{**} = \frac{2\alpha}{1+\alpha},$$
$$\hat{\hat{r}}(\alpha) \equiv \frac{1+\alpha}{\alpha} \ln \frac{1+\alpha}{1-\alpha}.$$

Gopalsamy and Liu in [3] conjectured that for $0 < \alpha < 1$,

$$r \le \hat{\hat{r}}(\alpha) \equiv \frac{1+\alpha}{\alpha} \ln \frac{1+\alpha}{1-\alpha} \tag{1.2}$$

is the necessary and sufficient condition of the global asymptotic stability for the positive equilibrium $N^* = \frac{1}{a+b}$. This conjecture has been solved affirmatively by Muroya and Kato in [2] for $0 < \alpha < 0.634817\cdots$ and by Li and Yuan

* Corresponding author.
*E-mail addresses:* lihx@mail.bnu.edu.cn (H. Li), ryuan@bnu.edu.cn (R. Yuan).

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in [1] for $0.625 < \alpha < 1$. Recently, Muroya et al. in [4] considered the case $-b < a < 0$ and established that for $-0.72534\cdots \le \alpha < 0$, (1.2) is the necessary and sufficient condition for the global asymptotic stability of the unique equilibrium $N^* = \frac{1}{a+b}$ (this paper was submitted). They conjectured that (1.2) is the necessary and sufficient condition for the global asymptotic stability of the positive equilibrium as $\alpha \in (-1, 0)$ which is called the extended Gopalsamy and Liu's conjecture. In this paper, using the method different from Muroya's, an affirmative answer is given to the extended Gopalsamy and Liu's conjecture for $-1 < \alpha < 0$. We formulate our main theorem as follows:

**Theorem 1.1.** *For $\alpha \in (-1, 0)$, $N^* = \frac{1}{a+b}$ is global asymptotic stability if and only if (1.2) holds.*

For the case $\alpha = 0$, Gopalsamy in [5] proved that the sufficient and necessary condition for the global asymptotic stability of $N^*$ is $r \le 2$. In [3] Gopalsamy and Liu proved that $N^*$ is global asymptotic stability independent with the delay when $\alpha \ge 1$.

For a thorough study of delay differential equations we can refer to [5–7]. Several authors have investigated the stability, oscillatory characteristics and chaotic behavior of Eq. (1.1) (see [2,5,8–16] and the references therein). For the existence of periodic solution and almost periodic solution of Eq. (1.1) in nonautonomous case we can refer to [17–21].

The organization of the paper is as follows. In Section 2, we introduce some previous results we need and establish some sufficient condition for the equilibrium to be globally asymptotically stable. We prove our main theorem in Section 3.

## 2. Lemmas

We recall the definition of solutions of Eq. (1.1) as follows. We say that a function $N : \mathbb{R}^+ \to \mathbb{R}$ is a solution of Eq. (1.1) if the following conditions are satisfied:

(1) $N$ is continuous on $\mathbb{R}^+$;
(2) the derivative $\dot{N}(s)$ of $N(s)$ exists everywhere, with the possible exception of the point $s = n(n \in \mathbb{N})$, where one-sided derivative exists;
(3) $N$ satisfies Eq. (1.1) on each interval $(n, n+1)$, $n \in \mathbb{N}$.

If $N(s)$ is a solution of Eq. (1.1), then we know that

$$N(s) = N(n)\exp\left\{r\int_n^s (1 - aN(\xi) - bN(n))\mathrm{d}\xi\right\}, \quad n \le s \le n+1,\ n = 0, 1, 2, 3, \ldots,$$

and $N(s) > 0$ if and only if $N(0) > 0$.

Let $z(t) = \frac{1}{N(t)}$ for $t \in [n, n+1]$. From Eq. (1.1), we know that

$$\frac{\mathrm{d}z(t)}{\mathrm{d}t} = (-r + rbN(n))z(t) + ra,$$

so for $t \in [n, n+1]$,

$$\begin{aligned} z(t) &= \mathrm{e}^{(-r+rbN(n))(t-n)}\left[\frac{1}{N(n)} + \int_n^t \mathrm{e}^{(r-rbN(n))(s-n)} rads\right] \\ &= \mathrm{e}^{r(-1+bN(n))(t-n)}\left[\frac{1}{N(n)} + ra\frac{\mathrm{e}^{r(1-bN(n))(t-n)} - 1}{r(1-bN(n))}\right] \end{aligned}$$

and

$$N(t) = \frac{N(n)\mathrm{e}^{r(1-bN(n))(t-n)}}{1 + raN(n)\frac{\mathrm{e}^{r(1-bN(n))(t-n)}-1}{r(1-bN(n))}}.$$

Using the same notation as [2], we introduce some known results. Assume that

$$\begin{cases} 1 + aN(n)\dfrac{\exp\{r(1-bN(n))\} - 1}{1 - bN(n)} > 0, & \text{if } b = 0, \text{ or } b \ne 0 \text{ and } N(n) \ne \dfrac{1}{b}; \\ 1 + aN(n)r > 0, & \text{if } b \ne 0 \text{ and } N(n) = \dfrac{1}{b}. \end{cases}$$

Then

$$N(n+1) = \begin{cases} \dfrac{N(n)\exp\{r(1-bN(n))\}}{1 + aN(n)\frac{\exp\{r(1-bN(n))\}-1}{1-bN(n)}}, & \text{if } b = 0, \text{ or } b \ne 0 \text{ and } N(n) \ne \dfrac{1}{b}; \\ \dfrac{N(n)}{1 + aN(n)r}, & \text{if } b \ne 0 \text{ and } N(n) = \dfrac{1}{b}. \end{cases} \tag{2.1}$$