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Existence results of periodic solutions for non-autonomous differential delay equations with asymptotically linear properties

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ABSTRACT

In this paper, we study a class of non-autonomous differential delay equations which can be changed to Hamiltonian systems. By estimating Maslov-type index of the related Hamiltonian systems at infinity and at origin, we establish the existence of periodic solutions of the differential delay equations.

1. Introduction and main result

Since the work of Jones [1] in 1962, many authors studied the following two autonomous differential delay equations

$$x'(t) = -f(x(t-\tau)), \tag{1.1}$$

and

$$x'(t) = -f(x(t-\tau_1)) - f(x(t-\tau_2)), \tag{1.2}$$

and their similar ones, where f is odd and continuous, τ, τ_1 and τ_2 are positive constants. A lot of remarkable results have been obtained by applying various methods, such as fixed point method and coincidence degree theory. For these results and methods, see [2–15]. Since it is not easy to construct a proper variational functional for differential delay equations, variational approaches have not been used to study the existence of periodic solutions for differential delay equations like (1.1) and (1.2) for a long time.

In [2], Kaplan and Yorke introduced a reduction method to study periodic solutions of (1.1) and (1.2) with $\tau = \tau_1 = 1$. The reduction method enables us to reduce the search for periodic solutions of (1.1) and (1.2) and their similar ones to the problems of finding periodic solutions for related systems of ordinary differential equations. By Kaplan and Yorke's method, variational approaches can be used to study the related systems (see [6–9]). In recent years, the authors in [14] and [15] considered (1.1) and the following second order differential delay equation

$$x''(t) = -f(x(t-\tau)) \tag{1.3}$$

by variational methods directly, respectively. That is to say that they do not necessarily transform the existence problem of (1.1) or (1.3) to existence problems for related systems. But we find that their method does not apply to differential delay

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equations with more than one delay such as (1.2), or to non-autonomous cases. In fact, the methods in [6–9] also cannot used to study the non-autonomous cases, since the functionals used there are not S^1-invariant in non-autonomous cases.

Motivated by the work in [6–9,14,15], in this paper we are concerned with the existence of periodic solutions of the following two non-autonomous differential delay equations

$$x'(t) = -f(t, x(t-\tau)), \tag{1.4}$$

and

$$x'(t) = -g(t, x(t-\tau_1)) - g(t, x(t-2\tau_1)), \tag{1.5}$$

where $\tau = 1/4, \tau_1 = 1/6, f$ and g are odd with respect to x. We first use Kaplan and Yorke's reduction technique to reduce (1.4) and (1.5) to related Hamiltonian systems. Then Maslov-type index (or Conley–Zehnder index) introduced and developed in [16–18] is applied to study the existence of periodic solutions of the related Hamiltonian systems. To the best of the author's knowledge, Maslov-type index is the first time to be used to study the existence of periodic solutions for differential delay equations.

Before introducing our main result, let us state some assumptions on f and g. Throughout the present paper, we suppose that the following conditions hold.

(H1) $f(t, x) \in C(\mathbb{R} \times \mathbb{R}, \mathbb{R})$ is odd with respect to x, τ periodic in t and satisfies

$$\lim_{x\to 0} \frac{f(t,x)}{x} = \alpha_0(t), \qquad \lim_{x\to\infty} \frac{f(t,x)}{x} = \alpha_\infty(t),$$

uniformly in t.

(H2) $g(t, x) \in C(\mathbb{R} \times \mathbb{R}, \mathbb{R})$ is odd with respect to x, τ_1 periodic in t and satisfies

$$\lim_{x\to 0} \frac{g(t,x)}{x} = \beta_0(t), \qquad \lim_{x\to\infty} \frac{g(t,x)}{x} = \beta_\infty(t),$$

uniformly in t.

(H3) Write $\bar{\alpha}_0 = \frac{1}{\tau}\int_0^\tau \alpha_0(t)\mathrm{d}t$, $\bar{\alpha}_\infty = \frac{1}{\tau}\int_0^\tau \alpha_\infty(t)\mathrm{d}t$. Let $|\bar{\alpha}_0 - \bar{\alpha}_\infty| \geq 2\pi$ and $\bar{\alpha}_0, \bar{\alpha}_\infty \neq 2k\pi$ for $k \in \mathbb{Z}$.

(H4) Write $\bar{\beta}_0 = \frac{1}{\tau_1}\int_0^{\tau_1} \beta_0(t)\mathrm{d}t$, $\bar{\beta}_\infty = \frac{1}{\tau_1}\int_0^{\tau_1} \beta_\infty(t)\mathrm{d}t$. Let $|\bar{\beta}_0 - \bar{\beta}_\infty| \geq \pi$ and $\bar{\beta}_0, \bar{\beta}_\infty \neq 2k\pi$ for $k \in \mathbb{Z}$. Then our main results read as follows.

Theorem 1.1. *Assume that the conditions* (H1) *and* (H3) *hold. Then* (1.4) *possesses at least one nontrivial 1-periodic solution* $x(t)$ *with* $x(t) = -x(t - 2\tau)$.

Theorem 1.2. *Assume that the conditions* (H2) *and* (H4) *hold. Then* (1.5) *possesses at least one nontrivial 1-periodic solution* $x(t)$ *with* $x(t) = -x(t - 3\tau_1)$.

Remark 1.1. Here f does not necessarily satisfy those conditions such as $xf(x) > 0$ for $x \neq 0$ which was assumed and played an important role in [2,6,7,10], and $\pm(\int_0^x f(y)\mathrm{d}y - \frac{1}{2}\alpha_\infty x^2) > 0$ for $|x| > 0$ being small which was needed in [8,9].

Remark 1.2. Let $\bar{\alpha}_0 \in (0, 2\pi)$ and $\bar{\alpha}_\infty \in (-2\pi, 0)$. By a direct computation, we have $\upsilon = 0$, where υ is defined in [14] to denote the number of the periodic solutions of (1.1). Then by the main result in [14], one cannot obtain any periodic solution of (1.1). However, if we take $\alpha_0(t)$ and $\alpha_\infty(t)$ such that $\bar{\alpha}_0 = \frac{3}{2}\pi$ and $\bar{\alpha}_\infty = -\frac{3}{2}\pi$, then by Theorem 1.1, (1.4) and also (1.1) have at least one nontrivial periodic solution, respectively.

The paper is organized as follows. In Section 2 we give a brief introduction for Maslov-type index for symplectic paths which was introduced and developed in [16–18]. In Section 3 the proof of Theorem 1.1 will be carried out and the proof of Theorem 1.2 will be shown in Section 4.

2. Maslov-type index for symplectic paths

In this section, we give a brief introduction for Maslov-type index for symplectic paths which will be useful for our discussion.

For $N \in \mathbb{Z}^+$, let $W = Sp(N, \mathbb{R}) = \{M \in \mathcal{L}(\mathbb{R}^{2N}) | M^\top JM = J\}$ be the symplectic group, where $J = \begin{pmatrix} 0 & -I_N \\ I_N & 0 \end{pmatrix}$ is the standard symplectic matrix with I_N being the identity matrix in $\mathbb{R}^N$. Let $W^* = \{M \in W | 1 \notin \sigma(M)\}$ and $W^0 = W \setminus W^*$. Denote by $A(t)$ a symmetric and continuous 1-periodic matrix in $\mathbb{R}^{2N}$. Then the fundamental solution matrix of the following linear Hamiltonian system with 1-periodic coefficient matrix

$$y'(t) = JA(t)y \tag{2.1}$$

belongs to the set

$$P = \{\gamma \in C^1([0, 1], W) | \gamma(0) = I, \gamma'(1) = \gamma'(0)\gamma(1), J\gamma'(t)\gamma^{-1}(t) \text{ is symplectic}\},$$