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Exponential stability of equilibria of differential equations with time-dependent delay and non-Lipschitz nonlinearity[☆]

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ABSTRACT

This paper studies stability of equilibria of differential equations with time-dependent delay and non-Lipschitz nonlinearity. For this class of problems, we develop a novel method of analysis, the relative nonlinear measure method. Using this method, we obtain a sufficient condition for exponential stability. Moreover, this condition is used to study the stability of the equilibrium of a neural network model. Finally, some examples illustrate that our results are improvement and extension of some existing ones.

1. Introduction

In this paper, we discuss exponential stability of differential equations with the form

$$\begin{aligned} &\frac{\mathrm{d}u(t)}{\mathrm{d}t} = F(u(t)) + G(u(t-\tau(t))), \quad t \geq t_0 \\ &u(t) = \phi(t), \quad t \in [t_0 - b, t_0] \end{aligned} \tag{1.1}$$

where $t_0 \geq 0$ and $b > 0$ are constants, F and G are nonlinear partially Lipschitz continuous operators from an open subset Ω of $\mathbb{R}^n$ into $\mathbb{R}^n$, $u(t) \in \Omega$ for $t \geq t_0$, the delay function $\tau(t)$ satisfies $0 \leq \tau(t) \leq b$ for $t \geq t_0$, and $\phi(\cdot) \in C([t_0 - b, t_0], \Omega)$ is an initial function with the norm $\|\phi\| = \sup_{t_0 - b \leq s \leq t_0} \|\phi(s)\|$, here $C([t_0 - b, t_0], \Omega)$ denotes the space of all continuous functions from $[t_0 - b, t_0]$ into Ω.

Stability analysis of the delay differential equations is important for many problems in applications. Many excellent results were obtained in this area [1–14]. Some such results are obtained using Lyapunov functions [1,4,10]. Constructing a Lyapunov function may be a difficult task. Besides, some existing results rely on restrictive conditions on the coefficients of the problem such as Lipschitz continuity [8], strict monotonicity [11], and boundedness [7,14]. This paper presents a new method of stability analysis for (1.1) with minimal assumptions on F and G. The latter are only required to be partially Lipschitz continuous and delay $\tau(t)$ is assumed to be bounded. Our method also does not require constructing a Lyapunov function. The method is used to obtain the sufficient condition for exponential stability of the equilibria of (1.1), which provides estimates for the exponential decay of solutions. We apply these results to study a class of neural networks with time-dependent delays.

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The paper is arranged as follows. Section 2 presents exponential stability results of equilibria of (1.1). In Section 3, we apply these results to a class of neural networks with time-varying delays. Conclusions are drawn in Section 4.

2. Main results

We start by introducing some notations, definitions and basic results that will be employed throughout the paper.

$\mathbb{R}^n$ is endowed with the l^1-norm $\|\cdot\|_1$ defined by $\|x\|_1 = \sum_{i=1}^n |x_i|$ for every $x = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n$. Ω is an open subset in $\mathbb{R}^n$ and Ω_i denotes the projection of the subset Ω on the ith axis of $\mathbb{R}^n$.

Definition 1. The function $h : \Omega_i \to \mathbb{R}$ is said to be partially Lipschitz continuous on Ω_i, if for any $x \in \Omega_i$ there exists a constant $L_x > 0$ such that

$$|h(y) - h(x)| \le L_x |y - x|, \quad \forall y \in \Omega_i.$$

The constant

$$L_{\Omega_i}(h, x) = \sup_{y \in \Omega_i, y \neq x} \frac{|h(y) - h(x)|}{|y - x|}$$

is called minimal partial Lipschitz constant of h with respect to x on Ω_i. Particularly, if $\Omega_i = \mathbb{R}$, then the function h is called partially Lipschitz continuous.

The operator $f = (f_1, f_2, \ldots, f_n) : \Omega \to \mathbb{R}^n$ is partially Lipschitz continuous on Ω if each function f_i is partially Lipschitz continuous on Ω_i.

Remark 1. It is obvious that each Lipschitz continuous function is partially Lipschitz continuous. However, some partially Lipschitz continuous functions may not be Lipschitz continuous.

Definition 2 (*[15]*). Let f be an operator form Ω to $\mathbb{R}^n$ and $x^0 \in \Omega$. Then

(1) the constant

$$m_\Omega(f) = \sup_{x,y \in \Omega, x \neq y} \frac{\langle f(x) - f(y), \operatorname{sgn}(x - y)\rangle}{\|x - y\|_1}$$

is called the nonlinear measure of f on Ω;

(2) the constant

$$m_\Omega(f, x^0) = \sup_{x \in \Omega, x \neq x^0} \frac{\langle f(x) - f(x^0), \operatorname{sgn}(x - x^0)\rangle}{\|x - x^0\|_1}$$

is called the relative nonlinear measure of f at x^0 on Ω. Here $\langle \cdot, \cdot \rangle$ denotes the inner product and $\operatorname{sgn}(x) = (\operatorname{sgn}(x_1), \operatorname{sgn}(x_2), \ldots, \operatorname{sgn}(x_n))^T$ denotes the sign vector of $x \in \mathbb{R}^n$, where $\operatorname{sgn}(r)$ is the usual sign function of any $r \in \mathbb{R}$.

Remark 2. The constants $m_\Omega(f)$ and $m_\Omega(f, x^0)$ are allowed to be infinite. However, if f is Lipschitz continuous on Ω, then $m_\Omega(f) < \infty$. If f is partially Lipschitz continuous with respect to x^0 on Ω, then $m_\Omega(f, x^0) < \infty$.

It is useful to notice that, for any point $x \in \mathbb{R}^n$,

$$\begin{cases} \|x\|_1 = \langle x, \operatorname{sgn}(x)\rangle & \text{and} \\ \|x\|_1 \ge \langle x, \operatorname{sgn}(y)\rangle & \text{for all } y \in \mathbb{R}^n. \end{cases} \tag{2.1}$$

Definition 3. Let x^* be an equilibrium point of the system (1.1) and Ω an open neighborhood of x^*. We say that x^* is exponentially stable on Ω if there exist two positive constants α and M such that

$$\|u(t) - x^*\|_1 \le M e^{-\alpha(t - t_0)} \sup_{t_0 - b \le s \le t_0} \|\phi(s) - x^*\|_1, \quad t \ge t_0,$$

where $u(t)$ is the solution of the system (1.1) initiated from the function $\phi(\cdot) \in C([t_0 - b, t_0], \Omega)$.

Moreover, if $\Omega = \mathbb{R}^n$, i.e., x^* is exponentially stable on the whole space $\mathbb{R}^n$, the system (1.1) is said to be globally exponentially stable.

Proposition 1. *The solutions of the time-delayed system (1.1) exist in the global time interval* $[t_0, \infty)$.

Proof. The solutions of the time-delayed system (1.1) locally exist [16]. Then the system (1.1) enjoys a solution $x(t, \phi)$ satisfying $x(t_0, \phi) = \phi$ for $t \in [t_0, t^*(\phi))$ where $t^*(\phi) \in (t_0, +\infty)$ or $t^*(\phi) = +\infty$ such that $[t_0, t^*(\phi))$ is the maximal right existence interval of the solution $x(t, \phi)$. Let $T_0 \in (t_0, \infty)$ be any finite time such that $x(t, \phi)$ is a solution of the system (1.1) for $t \in [t_0, T_0)$. Since F, G is partially Lipschitz continuous, there exist constants $L_\Omega(F, 0), L_\Omega(G, 0) > 0$ such that

$$\|F(u(t)) - F(0)\| \le L_\Omega(F, 0)\|u(t)\| \quad \text{and} \quad \|G(u(t)) - G(0)\| \le L_\Omega(G, 0)\|u(t)\|$$