Nonlinear Analysis: Real World Applications 11 (2010) 3704–3710

Contents lists available at ScienceDirect

Nonlinear Analysis: Real World Applications

journal homepage: www.elsevier.com/locate/nonrwa

Exponential stability criterion for chaos synchronization in modulated time-delayed systems

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ARTICLE INFO

Article history:
Received 17 November 2009
Accepted 27 January 2010

Keywords:
Chaos synchronization
Exponential stability
Delayed system
Modulated delay time
Krasovskii–Lyapunov theory

ABSTRACT

In this paper, we consider two unidirectionally coupled time delayed systems with periodic delay time modulation. A new stability condition for synchronization is derived analytically with the help of the Krasovskii–Lyapunov approach for single and two time delays. The numerical calculations greatly support our analytical results.

1. Introduction

The general definition of the complete synchronization phenomenon is as follows: the trajectories of the drive and response systems are identical notwithstanding they start from different initial conditions. However, a slight deviation of the initial conditions, for chaotic dynamical systems, will lead to completely different trajectories [1–4]. Different synchronization criteria has been well studied for discrete and continuous systems [5–18].

One of the most important applications of chaos synchronization is in Cryptography [19,20]. It is proved that low dimensional chaotic systems do not ensure a sufficient level of security for communications, as the associated chaotic attractors can be reconstructed with some effort and the hidden message can be retrieved by an attacker. One way to overcome this problem is by considering a high dimensional system with more than one positive Lyapunov exponents. In this regard, the time-delayed system received a lot of attention; i.e. $\dot{x} = f(x(t), x(t-\tau))$ where τ is a delay time. With increases in time-delay τ, the system is more complex and the number of positive Lyapunov exponents increase and the system eventually transits to hyperchaos. For this reason, a time-delayed system was considered as a suitable candidate for communication systems. If the delay time τ is known, the time-delayed system becomes quite simple and the message encoded by the chaotic signal can be extracted by the common attack methods [21]. If we consider a time-delayed system in which the delay time is not constant but modulated in time, then the communication system will be more secure. This is the practical significance of the study of chaos synchronization in time delayed systems with modulated delay time.

It is important for the receiver in secure communication to synchronize quickly with the sender; i.e. it is practical for the error system to have an exponential convergence rate. In this paper, we obtain the criterion of exponential stability for the error system and estimate the corresponding exponential convergence rate. In absence of exponential convergence rate, exponential stability was converted to asymptotical stability.

This paper is organized as follows. In Section 2 we consider two coupled chaotic Ikeda systems [22] with one way linear feedback coupling. The chaotic properties and the synchronization phenomenon are investigated numerically for

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doi:10.1016/j.nonrwa.2010.01.015

different k. Sections 3 and 4 are based on analytic derivations of a new kind of sufficient stability condition for one and two time delay respectively. The coupled systems can be useful for cryptographic encoding [23]. Finally in Section 5, we have summarized our results.

2. Ikeda model with one delay

We consider chaos synchronization between two unidirectionally coupled time delayed Ikeda systems. The driver system reads

$$\dot{x} = -ax + m_1 \sin(x(t-\tau_1)) \tag{1a}$$

and the response system reads

$$\dot{y} = -ay + m_2 \sin(y(t-\tau_2)) + k(x-y) \tag{1b}$$

where k is the coupling rate between the driver x and the response system y. Physically x is the phase lag of the electric field across the resonator, a is the relaxation coefficient for the dynamical variable, and $m_{1,2}$ are the laser intensities injected into the systems. τ_1 and τ_2 are the round-trip times of the light in the resonators or feedback delay times in the coupled systems [24]. The Ikeda model was introduced to describe the dynamics of an optical bistable resonator and is well known for delay-induced chaotic behavior [22].

The system (1a) is chaotic for the parameter values $a = 1.0$, $m_1 = 6.0$, $\tau_1 = 1.0$. We wish to replace the time-delay parameter τ_1 as a function of time instead of the constant time delay, in the form [25,26],

$$\tau_1 = a_0 + a_1 \sin(\omega_1 t) \tag{2}$$

where a_0 is the zero frequency component, a_1 is the amplitude, and $\omega_1/2\pi$ is the frequency of the modulation. For $k = 0$ the system is chaotic at $a_0 = 1.0$, $a_1 = 0.5$ and $\omega_1 = 5$.

3. Coupled system and stability conditions

Eqs. (1a) and (1b) allow for the synchronization manifold $x = y$. The necessary condition for synchronization of the system (1) is $m_1 = m_2$ with $\tau_1 = \tau_2$. We denote the error signal as $\Delta = x - y$. Then from Eqs. (1a) and (1b), we can derive the dynamics of the error

$$\dot{\Delta}(t) = -(a+k)\Delta(t) - m_1 \cos(x_{\tau_1})\Delta(t-\tau_1) \tag{3}$$

it is obvious that $\Delta = 0$ is a trivial solution of Eq. (3). To study the stability of the synchronization manifold $x = y$, we can use a Krasovskii–Lyapunov functional approach.

In general, the precise presentation of error system (3) is often so complex that it is very difficult to analyze its stability. We consider the error system as

$$\dot{\Delta}(t) = -r(t)\Delta(t) + s(t)\Delta(t-\tau(t)), \tag{4}$$

with initial condition $\Delta(t) = \Phi(t)$, $t \in [-\tau, 0]$ where Φ is real-value continuous function on $[-\tau, 0]$. It is well known that it is vital for the receiver in secure communication to synchronize quickly with the sender; i.e. it is practical for system (4) to have an exponential convergence rate. In this paper, we obtain the criteria of exponential stability for the error system with variable delay time and estimate the corresponding exponential convergence rate.

Definition. If there exist $\alpha > 0$ and $r(\alpha) > 1$ such that

$$\|\Delta(t)\| \le r(\alpha)\mathrm{e}^{-\alpha t} \sup_{-\tau\le\theta\le 0} \|\Delta(\theta)\| \quad \forall\, t > 0,$$

then the system (4) is said to be exponentially stable, where α is called the exponential convergence rate, and the notation $\|\cdot\|$ denotes the Euclidian norm of a vector or a square matrix.

We can derive the following lemma for system (4).

Lemma 1. *Suppose that there exist three positive numbers p, q and α such that*

$$2r(t)p - 2\alpha p - q - \frac{p^2 s^2(t)\mathrm{e}^{2\alpha\tau(t)}}{q(1-\tau'(t))} > 0 \tag{5}$$

where $\tau'(t) = \mathrm{d}\tau(t)/\mathrm{d}t$, then system (4) is exponentially stable. Moreover

$$\|\Delta(t)\| \le \left[1 + \frac{q}{2\alpha p}(1 - \mathrm{e}^{-2\alpha\tau(t)})\right]^{1/2} \|\Phi\|\mathrm{e}^{-\alpha t}, \quad \forall t > t_0$$

where t_0 is a scalar such that $\Delta(t) \to 0$ after $t > t_0$.