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# Hopf bifurcation analysis on an Internet congestion control system of arbitrary dimension with communication delay

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ABSTRACT

This paper carries out a Hopf bifurcation analysis on a model of Internet congestion control system for a network with arbitrary topology. The general form of the rate-based Kelly model for a multi-source multi-link network with a communication delay is considered. Assuming the communication delay as a bifurcation parameter, we find that when the delay parameter passes a critical value, a periodic solution bifurcates from the equilibrium point. The stability and direction of bifurcating periodic solutions are studied by using the center manifold theorem and the normal form theory. We simulate our model for a typical example to show the applicability of the approach.



## 1. Introduction

Congestion control has become one of the major problems of the Internet today due to the sustained explosive growth of subscribers over the last decade. The basic aim of the Internet congestion control algorithm is to adjust the sending flow rates of end hosts in order to avoid congestion at network links. The whole congestion control system employs a combination of the end-to-end congestion control mechanism of the transmission control protocol (TCP) at sources, and the active queue management (AQM) algorithm at routers. The TCP congestion control mechanism halves the congestion window for every packet loss, and increases the congestion window one segment per Round Trip Time (RTT) otherwise [1]. The AQM mechanism provides feedback information indicating the congestion level at each router by either dropping or marking packets. Some examples of AQM mechanisms are Random Early Detection (RED) as the most common AQM algorithm [2], Random Early Marking (REM) [3], and Adaptive Virtual Queue (AVQ) [4].

Considerable progress in mathematical modeling of congestion control system in the past decade has motivated researchers to study theoretic aspects in the system behavior, such as stability and robustness of the Internet congestion control system. Internet congestion control is a feedback system modeled by a nonlinear delay differential equation (NDDE) [5–7] and may perform complicated behaviors such as bifurcation and chaos, when the system loses its stability [8–12]. The complicated behavior in the network congestion control system has been studied since the work of Veres and Boda in [8], where it was demonstrated that TCP congestion control can be chaotic in certain circumstances. Subsequent studies investigated bifurcation and chaos in models of the TCP/AQM system [13–25]. Ranjan, et al. [13], using a linear stability analysis for a nonlinear first-order discrete-time model, demonstrated that bifurcation occurs as system parameters are varied. The paper also described the detrimental consequences of oscillatory or chaotic behavior on the performance of TCP sources. La [14] extended this work for a tandem network with two bottleneck links. By changing the system parameters only at one of the bottlenecks while fixing the parameters at the other, he demonstrated the propagation of instability.

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Furthermore, he showed that the local conditions for stability based on single node analysis were not sufficient for general network stability.

Several Hopf bifurcation analyses have been carried out in recent years [15–25]. Li, et al. [15] studied the conditions for Hopf bifurcation in a rate control algorithm and determined the stability of bifurcating periodic solutions and the direction of Hopf bifurcation by applying the normal form theory and the center manifold theorem. They considered a single link and single source network modeled by a first-order NDDE and proved the occurrence of Hopf bifurcation by varying the system gain. In addition, in [16] the Hopf bifurcation was studied in a REM algorithm described by the second order NDDE when the feedback delay passes through a critical value. Delay induced Hopf bifurcation in a one-order model of the congestion control system was analyzed in [17] and the critical value of delay, as a bifurcation parameter, was obtained for any fixed value of the system gain. Also, Raina [18] studied local stability of a NDDE model of TCP/AQM for a single-source single-link network and derived explicit conditions for the onset of stable limit cycles. A fluid model of TCP with an approximation of Drop Tail was also analyzed using bifurcation theory [19]. In [20], the Hopf bifurcation was studied in a single-source single-link network implementing the TCP protocol at the source and the Exponential RED (E-RED) algorithm at the link. Similar results were also obtained in [21] for a single-link and two-source TCP/E-RED network model. Recently, Ding et al. carried out a Hopf bifurcation analysis for fluid flow models of TCP/AQM networks [22] and a delay induced Hopf bifurcation analysis on a dual model of the Internet congestion control algorithm [23].

Most of the previous analyses involve simple network topologies such as single-source single-link. Therefore, it is a significant open challenge to study Hopf bifurcation for nonlinear delayed models of the general network topology with an arbitrary number of sources and links. In our recent works, Hopf bifurcation in congestion control system was studied for a two-source two-link network [24,25]. We considered the vector of gain parameters [24] and the delay [25] as bifurcation parameters. We demonstrated that the Hopf bifurcation occurs when the parameters pass through a critical value.

The present paper extends the work of [17,25] for a more general model of a multi-link multi-source homogeneous network. In this paper, rather than considering the system gain, which is a design parameter, we choose delay as a bifurcation parameter. Due to the variation of delay depending on the network congestion status, we show that the system may exhibit complex dynamic behavior. The paper presents a Hopf bifurcation analysis for real scalable networks to provide a useful framework for future control schemes. The critical delay parameter depends on some system parameters such as the gain of the system, which can be selected as a control parameter at sources, and the price function at routers. Using this analysis, one can adjust the system parameters in order to avoid undesirable oscillations due to the delay oscillations and obtain a satisfactory performance.

We consider Kelly's optimization framework for the rate allocation problem [5], and using a linear stability analysis, we derive the conditions for onset of Hopf bifurcation as the delay parameter varies. By applying the normal form theory and the center manifold theorem, the direction of Hopf bifurcation and the stability of the bifurcated solutions will be studied. A numerical example for a simple congestion control system is simulated to justify the theoretical results.

The paper proceeds as follows. In Section 2, a general model for the congestion control system with arbitrary topology is introduced. Then, we analyze the model and determine the conditions of Hopf bifurcation occurrence, in Section 3. Section 4 investigates the direction and the stability of Hopf bifurcated solution of the system. Simulation results are shown in Section 5. Finally, concluding remarks and a brief discussion are given in the last section.

## 2. Congestion control system model

Consider a communication network consisting of a set of $n$ sources, which utilize a set of $m$ links. The model has been previously introduced in [5] and has been extended in [6,7]. Each flow $j$ identifies a unique source–destination pair and is associated with a route $r_j$, which is the collection of links through which the information flows from the source to the destination. If the route uses link $l$, we write $l \in r_j$. Let $\boldsymbol{R}$ be routing matrix of dimension $m \times n$ indicating the set of routes where we have:

$$R_{lj} = \begin{cases} 1 & \text{if flow } j \text{ uses link } l \\ 0 & \text{otherwise.} \end{cases}$$

Assume that every user adopts rate-based flow control. Let $w_j$ and $x_j(t)$ denote the user's willingness to pay (per unit time) and its rate at time $t$, respectively. Now suppose that at time $t$ each link $l$ charges a price per unit flow of $\mu_l(t) = p_l(\sum_{j:l\in r_j} x_j(t))$, where $p_l(.)$ is a strictly increasing function of the total rate going through link $l$. Consider the system of differential equations

$$\frac{\mathrm{d}}{\mathrm{d}t} x_j(t) = \kappa_j \left[ w_j - x_j(t) \sum_{l\in r_j} \mu_l(t) \right]$$

where $\kappa_j$ is a gain parameter.

Each user adjusts its rate based on the feedback from the links' prices in the network to equalize its willingness to pay and the total price. The feedback from a link can also be interpreted as a congestion indicator affecting the rates of flow through the link.