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On weakly nonlinear evolution of convective flow in a passive mushy layer

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ABSTRACT

The problem of weakly nonlinear convective flow in a mushy layer, with a permeable mush–liquid interface and constant permeability, is studied under operating conditions for an experiment. A Landau type nonlinear evolution equation for the amplitude of the secondary solutions, which is based on the Landau theory and formulation for the Rayleigh, R, number close to its critical value, R_c, is developed. Using numerical and analytical methods, the solutions to the evolution equation are calculated for both supercritical and subcritical conditions. We found, in particular, that for $R < R_c$, the amplitude of the secondary solutions decays with time. For $R > R_c$, the tendency for chimney formation in the mushy layer increases with time. In addition, in such a supercritical regime, the basic flow is linearly unstable and we see the presence of steady flow for large values of time. These results suggest a possible slow transition to turbulence in such a flow system.

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1. Introduction

Convective flows, in a horizontal dendritic layer (also known as a mushy layer), during alloy solidification, are known to produce undesirable effects in the final form of the alloy [1]. Hence, in recent years, several theoretical and experimental studies [2–8] have been performed to study the convective chimney formation in the mushy layer. It is well known that convection in the chimneys causes a thin hair like structural defect called 'freckles' in the solidified alloy.

Several years ago, Worster [2,3] proposed the governing equations for a mushy layer, in the limit of infinite Prandtl number and asymptotically large mushy solute Rayleigh number, and performed linear stability analysis. He detected two modes of instability and concluded that the mushy layer mode is responsible for the freckle formation. Amberg and Homsy [9] introduced a simplified, single-layer model by assuming a small growth Peclet number, δ, and infinite Lewis number. They performed weakly nonlinear analysis and calculated a critical value of the combined parameter (mush permeability and solid fraction variations) for the transition from supercritical to subcritical rolls. Anderson and Worster [10] employed a weakly nonlinear analysis of a simplified mushy layer model that was proposed by Amberg and Homsy [9]. A near eutectic approximation was applied and the limit of large far-field temperature was considered. Such asymptotic limits allowed them to examine the dynamics of the mushy layer. They also considered the limit of large Stefan number, which enabled them to reach a domain for the existence of the oscillatory mode of convection. Okhuysen and Riahi [6,7] studied a weakly nonlinear analysis of buoyant convection in binary alloy solidification for a permeable mush–liquid interface. A number of assumptions, made in the previous theoretical studies [2,3,9,10], were not considered in their study. The most important result of their study was the prediction of a subcritical down-hexagonal pattern for the variable permeability case that corresponds to the smallest value of the Rayleigh number.

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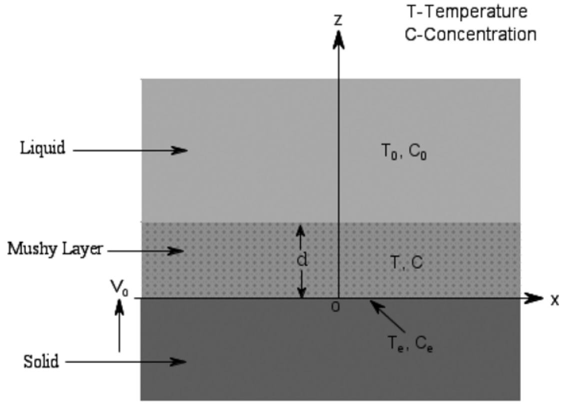

Fig. 1. Geometry.

The above mentioned works considered weakly nonlinear analysis of convection in a solidifying mushy layer. But none of these considered time evolution of the nonlinear flow and its effect on the chimney formation. In this regard, here for the first time, a Landau type nonlinear evolution equation for the amplitude of the secondary flow in a mushy layer is derived and the Landau coefficient is determined numerically. Previously several researchers [11,12] were able to study the nonlinear evolution system for a three-dimensional boundary layer and rotating disk flow problem.

2. Formulation and analysis

Here we consider a horizontal layer of binary alloy melt, of composition C_0 and temperature T_∞, which is cooled from below and for which the solidification front advances with a constant speed V_0 (Fig. 1).
The mushy layer is assumed to be in local thermodynamic equilibrium and the temperature T and composition C satisfy a linear relation [3]. Then the non-dimensional equations for momentum, continuity, heat and the solute for the dendrite region $(0 < z < \delta)$ are given by

$$K\vec{U} + \nabla P + R\Theta\hat{k} = 0, \tag{1a}$$

$$\nabla \cdot \vec{U} = 0, \tag{1b}$$

$$\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial z}\right)[\Theta - S\Phi] + \vec{U}\cdot\nabla\Theta = \nabla^2\Theta, \tag{1c}$$

$$\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial z}\right)[(1-\Phi)\Theta + C\Phi] + \vec{U}\cdot\nabla\Theta = 0. \tag{1d}$$

The corresponding boundary conditions are given by

$$\Theta + 1 = W = 0 \quad \text{at } z = 0, \tag{2a}$$

$$\Theta = \Phi = \partial_z W = 0 \quad \text{at } z = \delta, \tag{2b}$$

where $\vec{U} = U\hat{i} + V\hat{j} + W\hat{k}$ is the volume flux vector per unit area, which is also known as the Darcy velocity vector, U and V are the horizontal components of $\vec{U}$, W is the vertical component of $\vec{U}$ along the z-direction, P is the non-dimensional modified pressure, Θ is the non-dimensional temperature, or equivalently, composition, $\Theta = [T - T_L(C_0)]/\Delta T$, and also $\Theta = (C - C_0)/\Delta C$, where $\Delta T = T_L(C_0) - T_E$ and $\Delta C = C_0 - C_E$, T_E is the eutectic temperature, t is the time variable, Φ is the local solid volume fraction, $R = \beta\Delta C g \Pi_0/V\upsilon$ is the Rayleigh number, Π_0 is the reference value of the permeability of the porous medium, which is assumed to be finite, $K = \Pi_0/\Pi$, Π is the permeability of the medium, υ is the kinematic viscosity, β is the expansion coefficient of the solid, g is the acceleration due to gravity, $S = L/(\gamma\Delta T)$ is the Stefan number, γ is the specific heat per unit volume, L is the latent heat of solidification per unit volume, $C = (C_s - C_0)/\Delta C$ is a concentration ratio, C_s is the composition of the solid phase forming the dendrites, $\delta = \mathrm{d}V/\kappa$ is a growth Peclet number representing the dimensionless depth of the mushy layer; and $\hat{i}, \hat{j}$ and $\hat{k}$ are the unit vectors in the x-, y- and z-directions, respectively. Since we consider the permeability of the mushy layer to be constant for this present study, we assume $K = 1$.

The governing equations (1a)–(1d) and the corresponding boundary conditions (2a)–(2b) admit a basic motionless solution. Here the quantities with subscripts 'b' represent the base flow quantities, and θ, ϕ, $\vec{u}$, p represent the corresponding perturbed quantities.

$$\Theta = \theta_b(z) + \varepsilon\theta(x, y, z, t), \tag{3a}$$

$$\Phi = \phi_b(z) + \varepsilon\phi(x, y, z, t), \tag{3b}$$