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Helical flows of Maxwell fluid between coaxial cylinders with given shear stresses on the boundary

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ABSTRACT

Helical flows for a Maxwell fluid are studied between two infinite coaxial circular cylinders, at time $t = 0^+$; the inner cylinder begins to rotate around its axis and to slide along the same axis due to the torsional and longitudinal time dependent shear stresses. Exact solutions obtained with the help of finite Hankel transform and, presented under series form, satisfy all imposed initial and boundary conditions. The corresponding solutions for Newtonian fluid are also given as limiting cases. Finally, the influence of pertinent parameters – as well as a comparison between Maxwell and Newtonian fluids – on the velocity components and shear stresses is also analyzed by graphical illustrations.

1. Introduction

At present, numerical solutions to fluid mechanics problems are very attractive due to wide availability of computer programs. But these numerical solutions are insignificant if they cannot be compared with either analytical solutions or experimental results. Exact solutions are important not only because they are solutions of some fundamental flows, but also because they serve as accuracy standards for approximate methods, whether numerical, asymptotic or experimental. The first exact solutions corresponding to the motions of non-Newtonian fluids in cylindrical domains, seem to be those of Ting [1] for second grade fluids, Srivastava [2] for Maxwell fluids and Waters and King [3] for Oldroyd-B fluids. In the meantime, a lot of papers regarding such motions have been published [4–15]. However, most of them deal with motion problems in which the velocity field is given on the boundary. To the best of our knowledge, the first exact solutions for motions of non-Newtonian fluids due to a given shear stress on the boundary are those of Waters and King [16], Bandelli et al. [17] and Erdogan [18] over an infinite plate and Bandelli and Rajagopal [19] in cylindrical domains. However, little work has been done for motions of non-Newtonian fluids due to a shear stress given on the boundary [20–26] for motions in cylindrical domains.

The helical flow is of interest to both theoretical and practical points of view. The flow in an annular region between two coaxial circular cylindrical surfaces due to a combination of their rotation and the flow along the axis is known as helical flow, because, in general, the streamlines are helices [27]. Such a motion is very important to study the mechanism of viscoelastic fluids flow in many industry fields, such as oil exploitation, chemical and food industry, bio-engineering and lubrication studies [28]. Such a flow is of considerable basic interest because it includes as special cases, simple shear, channel, Couette, Poiseuille and pipe flows. The term helical flow was first introduced in 1956 by Rivlin [29] who derived

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the velocity distribution for fluids of the differential type in a concentric annular space. Coleman and Noll [30] also studied the concentric helical flow and gave the fundamental theory for a general fluid. Using Taylor series expansion of the velocity profiles, Wood [31] has considered the helical flow of an Oldroyd-B fluid due to the combined action of rotating cylinders and a constant pressure gradient. Fetecau et al. [32–37] studied some helical flows of Maxwell and Oldroyd-B fluids between two infinite coaxial cylinders and within an infinite cylinder by means of the expansion theorem of Steklov. The velocity fields and the associated tangential stresses are determined in form of series in terms of Bessel functions. More recently, work on helical flows for ordinary and fractional derivative models appear in [38–41].

The main goal of this note is to extend the results of Bandelli and Rajagopal [19, Sect. 4 and 5] to find some exact solutions to a new class of motions of Maxwell fluid. More exactly, our interest is to find the velocity field and the shear stress corresponding to the motion of Maxwell fluid between two infinite coaxial circular cylinders, with prescribed torsional and longitudinal time dependent shear stresses of the form Kt, on the boundary of inner cylinder. The general solutions, obtained by means of finite Hankel transforms and presented under series form in terms of Bessel functions $J_0(\bullet)$, $Y_0(\bullet)$, $J_1(\bullet)$, $Y_1(\bullet)$, $J_2(\bullet)$ and $Y_2(\bullet)$, satisfy all imposed initial and boundary conditions. The similar solutions corresponding to the Newtonian fluid appear as limiting cases. Finally, the solutions that have been obtained are compared by graphical illustrations and the influence of the pertinent parameters on the fluid motion is also analyzed by graphs.

2. Basic governing equations

The Cauchy stress $\mathbf{T}$ in an incompressible Maxwell fluid is given by [32–37]

$$\mathbf{T} = -p\mathbf{I} + \mathbf{S}, \qquad \mathbf{S} + \lambda(\dot{\mathbf{S}} - \mathbf{LS} - \mathbf{SL}^T) = \mu\mathbf{A}, \tag{1}$$

where $-p\mathbf{I}$ denotes the indeterminate spherical stress due to the constraint of incompressibility, $\mathbf{S}$ is the extra-stress tensor, $\mathbf{L}$ is the velocity gradient, $\mathbf{A} = \mathbf{L} + \mathbf{L}^T$ is the first Rivlin Ericksen tensor, μ is the dynamic viscosity of the fluid, λ is relaxation time, the superscript T indicates the transpose operation and the superposed dot indicates the material time derivative. The model characterized by the constitutive equations (1) contains as special case the Newtonian fluid model for $\lambda \to 0$. The model (1) is consistent with some important microscopically models of polymers and its predictions of the normal-stress differences are qualitatively acceptable. It has been quite useful in the study of dilute polymeric fluids in viscoelasticity. For the problem under consideration we shall assume a velocity field and an extra-stress of the form [2,4,42]

$$\mathbf{v} = \mathbf{v}(r,t) = w(r,t)\mathbf{e}_\theta + v(r,t)\mathbf{e}_z, \qquad \mathbf{S} = \mathbf{S}(r,t), \tag{2}$$

where $\mathbf{e}_\theta$ and $\mathbf{e}_z$ are unit vectors in the θ and z-directions of the cylindrical coordinate system r, θ and z. For such flows the constraint of incompressibility is automatically satisfied. If the fluid is at rest up to the moment $t = 0$, then

$$\mathbf{v}(r,0) = \mathbf{0}, \qquad \mathbf{S}(r,0) = \mathbf{0}, \tag{3}$$

and Eq. (1) implies $S_{rr} = 0$ and the meaningful equations [32]

$$\left(1 + \lambda\frac{\partial}{\partial t}\right)\tau_1(r,t) = \mu\left(\frac{\partial}{\partial r} - \frac{1}{r}\right)w(r,t), \qquad \left(1 + \lambda\frac{\partial}{\partial t}\right)\tau_2(r,t) = \mu\frac{\partial v(r,t)}{\partial r}, \tag{4}$$

where $\tau_1 = S_{r\theta}$ and $\tau_2 = S_{rz}$ are the shear stresses that are different of zero.

The balance of the linear momentum, in the absence of a pressure gradient in the axial direction and neglecting body forces, leads to the relevant equations ($\partial_\theta p = 0$ due to the rotational symmetry) [4,32]

$$\rho\frac{\partial w(r,t)}{\partial t} = \left(\frac{\partial}{\partial r} + \frac{2}{r}\right)\tau_1(r,t), \ \rho\frac{\partial v(r,t)}{\partial t} = \left(\frac{\partial}{\partial r} + \frac{1}{r}\right)\tau_2(r,t), \tag{5}$$

where ρ is the constant density of the fluid. Eliminating τ_1 and τ_2 between Eqs. (4) and (5) we attain the governing equations

$$\lambda\frac{\partial^2 w(r,t)}{\partial t^2} + \frac{\partial w(r,t)}{\partial t} = \nu\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\right)w(r,t); \quad r \in (R_1, R_2), t > 0, \tag{6}$$

$$\lambda\frac{\partial^2 v(r,t)}{\partial t^2} + \frac{\partial v(r,t)}{\partial t} = \nu\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)v(r,t); \quad r \in (R_1, R_2), t > 0, \tag{7}$$

where $\nu = \mu/\rho$ is the kinematic viscosity of the fluid.

The system of partial differential equations (6) and (7), with adequate initial and boundary conditions, can be solved in principle by several methods, their effectiveness strictly depending on the domain of definition. In our case the integral transforms technique presents a systematic, efficient and powerful tool. The Laplace transform can be used to eliminate the time variable while the finite Hankel transform can be employed to eliminate the spatial variable. However, the inversion procedure for the Laplace transform is heavy enough and needs much ingenuity. Here, we shall use the finite Hankel transforms.