Nonlinear Analysis: Real World Applications 11 (2010) 4342–4354

Contents lists available at ScienceDirect

Nonlinear Analysis: Real World Applications

journal homepage: www.elsevier.com/locate/nonrwa

# One-dimensional viscoelastic fluid model where viscosity and normal stress coefficients depend on the shear rate

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ARTICLE INFO

*Article history:*
Received 10 December 2008
Accepted 3 May 2010

*Keywords:*
One-dimensional model
Perturbed flows
Volume flow rate
Shear-dependent normal stress coefficients
Shear-dependent viscosity
Average pressure

ABSTRACT

We study the unsteady motion of a viscoelastic fluid modeled by a second-order fluid where normal stress coefficients and viscosity depend on the shear rate by using a power-law model. To study this problem, we use the one-dimensional nine-director Cosserat theory approach which reduces the exact three-dimensional equations to a system depending only on time and on a single spatial variable. Integrating the equation of conservation of linear momentum over the tube cross-section, with the velocity field approximated by the Cosserat theory, we obtain a one-dimensional system. The velocity field approximation satisfies both the incompressibility condition and the kinematic boundary condition exactly. From this one-dimensional system we obtain the relationship between average pressure and volume flow rate over a finite section of the tube with constant and variable radius. Also, we obtain the correspondent equation for the wall shear stress which enters directly in the formulation as a dependent variable. Attention is focused on some numerical simulation of unsteady/steady flows for average pressure, wall shear stress and on the analysis of perturbed flows.



## 1. Introduction

In this article, we study the unsteady flow of a non-Newtonian fluid in a uniform straight tube, modeled by a generalization of the second-order fluid using a hierarchical approach, called Cosserat theory. Let us consider the constitutive equation for viscoelastic fluids of differential type (also called Rivlin–Ericksen fluids) with complexity $n = 2$, given by (see e.g. [1])

$$\boldsymbol{T} = -p\boldsymbol{I} + \mu \boldsymbol{A}_1 + \alpha_1 \boldsymbol{A}_2 + \alpha_2 \boldsymbol{A}_1^2 \tag{1}$$

where $p$ is the pressure, $-p\boldsymbol{I}$ is the spherical part of the stress due to the constraint of incompressibility, $\mu$ is the coefficient of viscosity, and $\alpha_1, \alpha_2$ are the normal stress coefficients usually called normal stress moduli. The kinematical first two Rivlin–Ericksen tensors $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ are defined through (see [2])

$$\boldsymbol{A}_1 = \nabla \boldsymbol{\vartheta} + \left(\nabla \boldsymbol{\vartheta}\right)^T \tag{2}$$

and

$$\boldsymbol{A}_2 = \frac{\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{A}_1\right) + \boldsymbol{A}_1 \nabla \boldsymbol{\vartheta} + \left(\nabla \boldsymbol{\vartheta}\right)^T \boldsymbol{A}_1 \tag{3}$$

where $\boldsymbol{\vartheta}$ is the three-dimensional velocity field of the fluid and $\frac{\mathrm{d}}{\mathrm{d}t}(\cdot)$ denotes the material time derivative. In condition (3) the material time derivative of the tensor $\boldsymbol{A}_1$ is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{A}_1\right) = \frac{\partial}{\partial t}\left(\boldsymbol{A}_1\right) + \boldsymbol{\vartheta} \cdot \nabla \boldsymbol{A}_1.$$

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1468-1218/$ – see front matter 
doi:10.1016/j.nonrwa.2010.05.020

The thermodynamics and stability of the fluids related with the constitutive equation (1) have been studied in detail by Dunn and Fosdick [3], who showed that if the fluid is to be compatible with thermodynamics in the sense that all motions of the fluid meet the Clausius–Duhem inequality and the assumption that the specific Helmholtz free energy of the fluid is a minimum in equilibrium, then

$$\mu \geqslant 0, \quad \alpha_1 \geqslant 0, \quad \alpha_1 + \alpha_2 = 0. \tag{4}$$

Later, Fosdick and Rajagopal [4], based on the experimental observation, showed that for many non-Newtonian fluids of current rheological interest the reported values for $\alpha_1$ and $\alpha_2$ do not satisfy the restriction $(4)_{2,3}$, relaxed that assumption. Also, they showed that for arbitrary values of $\alpha_1+\alpha_2$, with $\alpha_1 < 0$, a fluid filling a compact domain and adhering to the boundary of the domain exhibits an anomalous behavior not expected on real fluids. The condition $(4)_3$ simplifies substantially the mathematical model and the corresponding analysis. The fluids characterized by (4) are known as second-grade fluids as opposed to the second-order fluids, i.e. fluids where $\alpha_1 + \alpha_2$ is arbitrary and $\alpha_1 < 0$. In the sequel we consider fluids with $\mu \geqslant 0, \alpha_1 < 0$ and $\alpha_1+\alpha_2$ is an arbitrary value. Also, it should also be added that the use of Clausius–Duhem inequality is the subject matter of much controversy (see e.g. [5]). Now, we consider an extension of the Rivlin–Ericksen fluid model (1) by introducing no constant viscosity and no constant normal stress moduli. This variables could in principle be a positive function of the principal invariants of the tensors $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$, but we will further assume that it is only depends on shear rate (see e.g. [6–8]). Taking into account the constitutive equation recently proposed by Massoudi and Vaidya [7] the Eq. (1), becomes

$$\boldsymbol{T} = -p\boldsymbol{I} + \mu\big(|\dot{\gamma}|\big)\boldsymbol{A}_1 + \alpha\big(|\dot{\gamma}|\big)\big(\alpha_1\boldsymbol{A}_2 + \alpha_2\boldsymbol{A}_1^2\big) \tag{5}$$

where

$$\mu\big(|\dot{\gamma}|\big) : \mathbb{R}^+ \to \mathbb{R}^+$$

is the shear-dependent viscosity function, and

$$\alpha\big(|\dot{\gamma}|\big) : \mathbb{R}^+ \to \mathbb{R}^+$$

is the shear-dependent normal stress coefficients function, where $\dot{\gamma}$ is a scalar measure of the rate of shear defined by $|\dot{\gamma}| = \sqrt{2\boldsymbol{D} : \boldsymbol{D}}$ with

$$\boldsymbol{D} := \frac{1}{2}\big(\nabla\boldsymbol{\vartheta} + \big(\nabla\boldsymbol{\vartheta}\big)^T\big)$$

being the rate of deformation tensor. It is seen that the Eq. (5) is similar to the models proposed by Man et al. (see [9,10]) and is in fact a generalization of these models as well. Furthermore, it has been argued that the model (5) captures several of the features of the higher grade type models, see [7]. Since the model under consideration is an ad hoc variation of a more rigorously derived model, there are no information yet on the thermodynamic restrictions of the material moduli. Experimental studies with polymers (see e.g. [11]), suspensions (see e.g. [12]) and liquid crystals (see e.g. [13]) indicate that for several fluids, one observes a substantial variation in viscosity and normal stress effects with the shear rate. This dependence is of a power-law type and experiments indicate that the tangential stress varies at a different rate with the shear rate than the normal stress, see [14]. The particular functional dependence of the viscosity (normal stress coefficients, respectively) on shear rate is generally chosen in order to fit experimental data and, in the case of a power-law fluid model, is given by

$$\mu(|\dot{\gamma}|) = k_1|\dot{\gamma}|^{n-1}, \qquad \alpha(|\dot{\gamma}|) = |\dot{\gamma}|^{m-1} \tag{6}$$

where the parameters $k_1$ and $n$, $m$ (positive constants) are called the consistency and the flow index, respectively. This model has the advantage that it reduces to the standard second-order fluid when $n = m = 1$ in (6) with $k_1 = \mu$, and can also account for shear-thinning and shear-thickening behavior of the fluid. This investigation may be found to be relevant in several physical, biological and engineering applications. The model (5), with condition (6), may also be pertinent in the study of blood flow in small vessels where elastic effects become prominent. As in the case of polymers, suspensions and liquid crystals, it is plausible that normal stress effect in blood is also dependent on the shear rate. A possible simplification to a three-dimensional model for an incompressible viscoelastic fluid inside a domain is to consider the evolution of average flow quantities using simpler one-dimensional models. Usually, in the case of flow in a tube, the classical one-dimensional models are obtained by imposing additional assumptions and integrating both the equations of conservation of linear momentum and mass over the cross-section of the tube. Here, we introduce a one-dimensional model based on the nine-director approach developed by Caulk and Naghdi [15]. This theory includes an additional structure of directors (deformable vectors) assigned to each point on a space curve (Cosserat curve), where a three-dimensional system of equations is replaced by a one-dimensional system depending on time and on a single spatial variable. The use of directors in continuum mechanics goes back to Duhem [16] who regards a body as a collection of points together with associated directions. Theories based on such a model of an oriented medium were further developed by the french scientist Eugène and François Cosserat [17] and have also been used by several authors in studies of rods, plates and shells (see e.g. [18–22]). An analogous hierarchical theory for unsteady/steady flows has been developed by Caulk and Naghdi [15] in straight tubes of circular cross-section and by Green and Naghdi [23] in channels. The same theory was applied to unsteady viscous fluid flow in curved tubes of circular and elliptic cross-section by Green et al. [24]. Recently, the nine-director theory has been applied to blood flow by Robertson and Sequeira [25] and by Carapau and Sequeira [26]. Also by Carapau et al. [27–30] and by Carapau [31–34] considering