Nonlinear Analysis: Real World Applications 11 (2010) 759–763

Contents lists available at ScienceDirect

Nonlinear Analysis: Real World Applications

journal homepage: www.elsevier.com/locate/nonrwa

A parabolic problem arising in Financial Mathematics

P. Amster [a], C. Averbuj [a], P. De Nápoli [a], M.C. Mariani [b,*]

[a] *FCEyN - Departamento de Matemática - Universidad de Buenos Aires, Argentina*

[b] *Department of Mathematical Sciences, New Mexico State University Las Cruces, NM 88003-0001, USA*

ARTICLE INFO

Article history:
Received 7 June 2007
Accepted 13 January 2009

Keywords:
Parabolic problems
Financial Mathematics

ABSTRACT

We study a parabolic problem arising in Financial Mathematics. Under suitable conditions, we prove the existence and uniqueness of solutions in a general domain using the method of upper and lower solutions and a diagonal argument.

Published by Elsevier Ltd

1. Introduction

In recent years there has been an increasing interest in problems arising in Financial Mathematics and in particular on option pricing. The standard approach to this problem leads to the study of equations of a parabolic type.

An option is a contract that gives the holder the right to trade in the future at a previously agreed price. A European *call option* is a right to buy a particular asset for an agreed amount at a specific time in future. A *put option* is the right to sell a particular asset for an agreed amount at a specific time in future.

In Financial Mathematics, usually the Black–Scholes model [1] is used to price these contracts, by means of a reversed-time parabolic partial differential equation. In this model, an important quantity is the so-called volatility. Volatility is a measure of the amount of fluctuation in the asset prices: a measure of randomness. It has a major impact on the value of the option; in mathematical terms, it corresponds to the diffusion coefficient in the Black–Scholes equation.

In the standard Black–Scholes model, a basic assumption is that the volatility is constant. Several models that have been proposed in recent years, however, allowed the volatility to be non-constant or a stochastic variable. For instance, in [2] a model with stochastic volatility is proposed. In this model the underlying security S follows, as in the standard Black–Scholes model, an stochastic process

$$\mathrm{d}S_t = \mu S_t \mathrm{d}t + \sigma_t S_t \mathrm{d}Z_t,$$

where Z is a standard Brownian motion. Unlike the classical model, the variance $v(t) = \sigma^2(t)$ also follows stochastic process given by

$$\mathrm{d}v_t = \kappa(\theta - v(t))\mathrm{d}t + \gamma\sqrt{v_t}\mathrm{d}W_t,$$

where W is another standard Brownian motion. The correlation coefficient between W and Z is denoted by ρ:

$$E(\mathrm{d}Z_t, \mathrm{d}W_t) = \rho \mathrm{d}t.$$

This leads to a generalized Black–Scholes equation:

$$\frac{1}{2}vS^2\frac{\partial^2 U}{\partial S^2} + \rho\gamma vS\frac{\partial^2 U}{\partial v\partial S} + \frac{1}{2}v\gamma^2\frac{\partial^2 U}{\partial v^2} + rS\frac{\partial U}{\partial S} + [\kappa(\theta - v) - \lambda v]\frac{\partial U}{\partial v} - rU + \frac{\partial U}{\partial t} = 0.$$

* Corresponding author.
E-mail address: mmariani@nmsu.edu (M.C. Mariani).

1468-1218/$ – see front matter. Published by Elsevier Ltd
doi:10.1016/j.nonrwa.2009.01.019

If we introduce the change of variables given by $y = \log S$, $x = \frac{v}{\gamma}$, $\tau = T - t$ the following problem for $u(x, y) = U(S, v)$ is obtained:

$$u_\tau = \frac{1}{2}\gamma x\left[\Delta u + 2\rho\frac{\partial^2 u}{\partial x\partial y}\right] + \frac{1}{\gamma}\left[\kappa(\theta - \gamma x) - \lambda\gamma x\right]\frac{\partial u}{\partial x} + \left(r - \frac{\gamma x}{2}\right)\frac{\partial u}{\partial y} - ru$$

in a cylindrical domain $\Omega \times (0, T)$, with $\Omega \subset \mathbb{R}^2$. A similar model has been considered in [3], for which the stationary equation has been studied in [4].

More general models with stochastic volatility have been considered for example in [5], where the following problem is derived from the Feynman–Kac relation:

$$\begin{cases} u_t = \frac{1}{2}Tr\left(M(x,\tau)D^2u\right) + q(x,\tau)\cdot Du, \\ u(x,0) = u_0(x) \end{cases}$$

for some diffusion matrix M and a payoff function u_0.

This discussion motivates us to consider the general parabolic problem

$$\begin{cases} Lu - u_t = g(u,x,t) & \text{in } \Omega\times(0,T) \\ u(x,0) = u_0(x) & \text{on } \Omega\times\{0\} \\ u(x,t) = h(x,t) & \text{on } \partial\Omega\times(0,T). \end{cases} \tag{1.1}$$

We shall assume that $\Omega \subset \mathbb{R}^d$ is an unbounded smooth domain, $g : [0, +\infty) \times \overline{\Omega} \times [0, T] \to [0, +\infty)$ is continuous and continuously differentiable with respect to u, L is a second order elliptic operator in non-divergence form, namely

$$Lu := \sum_{i,j=1}^{d} a^{ij}(x,t)u_{x_ix_j} + \sum_{i=1}^{d} b^i(x,t)u_{x_i} + c(x,t)u,$$

where the coefficients of L belong to the Hölder Space $C^{\delta,\delta/2}\left(\overline{\Omega}\times[0,T]\right)$ and satisfy the following conditions

$$\Lambda|v|^2 \ge \sum_{i,j=1}^{d} a^{ij}(x,t)v_iv_j \ge \lambda|v|^2 \quad (\Lambda \ge \lambda > 0)$$

$$|b^i(x,t)| \le C, \qquad c(x,t) \le 0.$$

Furthermore, we shall assume that $u_0 \in C^{2+\delta}(\overline{\Omega})$, $h \in C^{2+\delta,1+\delta/2}(\overline{\Omega}\times[0,T])$ and satisfy the following compatibility condition

$$h(x,0) = u_0(x) \quad \forall\, x \in \partial\Omega. \tag{1.2}$$

Our main result reads as follows:

Theorem 1.1. *Let L be the elliptic operator defined as above, and assume that $g(0, x, t) = 0$. Then for any $T > 0$ there exists $\theta_0 = \theta_0(\Lambda, d, \|b\|_\infty, T)$ such that if $\theta < \theta_0$, then for any initial and boundary conditions u_0 and h satisfying*

$$0 \le u_0(x) \le kT^{-\frac{d}{2}}e^{\frac{\theta}{T}|x|^2}$$

and

$$0 \le h(x,t) \le k(T-t)^{-\frac{d}{2}}e^{\frac{\theta}{T-t}|x|^2} \quad \text{for } x \in \partial\Omega,\ 0 \le t < T$$

for some constant k, there exists at least one solution u of the problem (1.1) satisfying

$$0 \le u(x,t) \le k(T-t)^{-\frac{d}{2}}e^{\frac{\theta}{T-t}|x|^2}.$$

We give a proof of Theorem 1.1 in Section 2, using the method of upper and lower solutions. We recall that u is called an upper (lower) solution of problem (1.1) if

$$\begin{cases} Lu - u_t \le (\ge) g(u,x,t) & \text{in } \Omega\times(0,T) \\ u(x,0) \ge (\le) u_0(x) & \text{on } \Omega\times\{0\} \\ u(x,t) \ge (\le) h(x,t) & \text{on } \partial\Omega\times(0,T). \end{cases}$$

On the other hand, we obtain a uniqueness result, which can be deduced immediately from the following version of the maximum principle.