Nonlinear Analysis: Real World Applications 12 (2011) 2435–2466

Contents lists available at ScienceDirect

Nonlinear Analysis: Real World Applications

journal homepage: www.elsevier.com/locate/nonrwa

Variational approach to complicated similarity solutions of higher-order nonlinear PDEs. II

V.A. Galaktionov [a], E. Mitidieri [b,*], S.I. Pohozaev [c]

[a] Department of Mathematical Sciences, University of Bath, Bath BA2 7AY, UK
[b] Dipartimento di Matematica e Informatica, Università di Trieste, Via A. Valerio 12/1 34127 Trieste, Italy
[c] Steklov Mathematical Institute, Gubkina St. 8, 119991 Moscow, Russia

ARTICLE INFO

Article history:
Received 25 February 2011
Accepted 1 March 2011

Keywords:
Quasilinear ODEs
Non-Lipschitz terms
Similarity solutions
Blow-up
Extinction
Compactons
Lusternik–Schnirel'man category
Fibering method

ABSTRACT

This paper continues the study that began in [1,2] of the Cauchy problem for $(x, t) \in \mathbb{R}^N \times \mathbb{R}_+$ for three higher-order degenerate quasilinear partial differential equations (PDEs), as basic models,

$$u_t = (-1)^{m+1} \Delta^m (|u|^n u) + |u|^n u,$$
$$u_{tt} = (-1)^{m+1} \Delta^m (|u|^n u) + |u|^n u,$$
$$u_t = (-1)^{m+1} [\Delta^m (|u|^n u)]_{x_1} + (|u|^n u)_{x_1},$$

where $n > 0$ is a fixed exponent and Δ^m is the $(m \geq 2)$th iteration of the Laplacian. A diverse class of degenerate PDEs from various areas of applications of three types: parabolic, hyperbolic, and nonlinear dispersion, is dealt with. General local, global, and blow-up features of such PDEs are studied on the basis of their *blow-up* similarity or *traveling wave* (for the last one) solutions.

In [1,2], the Lusternik–Schnirel'man category theory of variational calculus and fibering methods were applied. The case $m = 2$ and $n > 0$ was studied in greater detail analytically and numerically. Here, more attention is paid to a combination of a Cartesian approximation and fibering to get new compactly supported similarity patterns. Using numerics, such compactly supported solutions are constructed for $m = 3$ and for higher orders. The "smother" case of negative $n < 0$ is included, with a typical "fast diffusion–absorption" parabolic PDE:

$$u_t = (-1)^{m+1} \Delta^m (|u|^n u) - |u|^n u, \quad \text{where } n \in (-1, 0),$$

which admits *finite-time extinction* rather than blow-up. Finally, a homotopy approach is developed for some kind of classification of various patterns obtained by variational and other methods. Using a variety of analytic, variational, qualitative, and numerical methods allows us to justify that the above PDEs admit an infinite countable set of countable families of compactly supported blow-up (extinction) or traveling wave solutions.

1. Introduction: higher-order blow-up and compacton models

A general physical and PDE motivation for the present research can be found in [1,2], together with basic history and related key references, so we just briefly comment on where quasilinear elliptic problems under consideration are coming from.

* Corresponding author.
E-mail addresses: vag@maths.bath.ac.uk (V.A. Galaktionov), mitidier@units.it (E. Mitidieri), pokhozhaev@mi.ras.ru (S.I. Pohozaev).

1468-1218/$ – see front matter
doi:10.1016/j.nonrwa.2011.03.001

1.1. (I) Combustion-type models with blow-up

Our first model is a quasilinear degenerate 2*m*th-order parabolic equation of the reaction–diffusion (combustion) type:

$$u_t = (-1)^{m+1}\Delta^m(|u|^n u) + |u|^n u \quad \text{in } \mathbb{R}^N \times \mathbb{R}_+, \tag{1.1}$$

where $n > 0$ is a fixed exponent, $m \geq 2$ is integer, and Δ denotes the Laplace operator in $\mathbb{R}^N$. Physical, mathematical, and blow-up history of (1.1) for the standard classic case $m = 1$ and $m \geq 2$ is explained in [1,2, Section 1]. Consider *regional blow-up* solutions of (1.1)

$$u_S(x,t) = (T-t)^{-\frac{1}{n}} f(x) \quad \text{in } \mathbb{R}^N \times (0,T) \tag{1.2}$$

in separable variables, where $T > 0$ is the blow-up time. Then the similarity blow-up profile $f = f(x)$ solves a quasilinear elliptic equation of the form

$$(-1)^{m+1}\Delta^m(|f|^n f) + |f|^n f = \frac{1}{n} f \quad \text{in } \mathbb{R}^N. \tag{1.3}$$

This reduces to the following semilinear equation with a non-Lipschitz nonlinearity:

$$F = |f|^n f \Longrightarrow (-1)^{m+1}\Delta^m F + F - \frac{1}{n}|F|^{-\frac{n}{n+1}} F = 0 \quad \text{in } \mathbb{R}^N.$$

Scaling out the multiplier $\frac{1}{n}$ in the nonlinear term yields

$$F \mapsto n^{-\frac{n+1}{n}} F \Longrightarrow \boxed{(-1)^{m+1}\Delta^m F + F - |F|^{-\frac{n}{n+1}} F = 0 \quad \text{in } \mathbb{R}^N.} \tag{1.4}$$

For $N = 1$, this is a simpler ordinary differential equation (an ODE):

$$F \mapsto n^{-\frac{n+1}{n}} F \Longrightarrow \boxed{(-1)^{m+1}F^{(2m)} + F - |F|^{-\frac{n}{n+1}} F = 0 \quad \text{in } \mathbb{R}.} \tag{1.5}$$

According to (1.2), the elliptic problems (1.4) and the ODE (1.5) for $N = 1$ are responsible for the possible "geometrical shapes" of regional blow-up described by the higher-order combustion model (1.1).

Remark (*Relation to ODEs from Extended KPP Theory*)**.** There exists vast mathematical literature, starting essentially from the 1980s, devoted to the fourth-order ODEs (looking rather analogously to that in (1.5) for $m = 2$)

$$F^{(4)} = \beta F'' + F - F^3 \quad \text{in } \mathbb{R}, \tag{1.6}$$

where $\beta > 0$ is a parameter. This ODE also admits a complicated set of solutions with various classes of patterns and even with chaotic features. We refer to Peletier–Troy's book [3] for the most diverse account, as well as to papers [4,5], where a detailed and advanced solution description for (1.6) is obtained by combination of variational and homotopy theories. Regardless of their rather similar forms, the ODEs (1.5) belong to a completely different class of equations with *non-coercive* operators, unlike in (1.6). Therefore, direct homotopy approaches and several others, that used to be rather effective for (1.6), fail in principle for (1.5). In this sense, (1.5) is similar to the cubic ODE to be studied in Section 6:

$$\boxed{F^{(4)} = -F + F^3 \quad \text{in } \mathbb{R},} \tag{1.7}$$

of course, excluding complicated oscillatory behavior at finite interfaces, which are obviously nonexistent for analytic nonlinearities. However, we claim that the sets of solutions of (1.5), with $m = 2$, and of (1.7) are equivalent, though, not having a rigorous proof, we will devote some efforts to a homotopy approach connecting solutions of such smooth (analytic) and non-smooth ODEs. Thus, though going to develop homotopy approaches for classifying solutions of (1.5) (Section 4), our main tool to describe countable families of solutions $\{F_l\}$ is a combination of the Lusternik–Schnirel'man category-genus theory [6] and the fibering method [7,8]. Thus, we show that ODEs (1.5), as well as the PDE (1.4), admit infinitely many countable families of compactly supported solutions, and the whole solution set exhibits certain *chaotic* properties. Our analysis will be based on a combination of analytic (variational and others), numerical, and some more formal techniques. Explaining existence, multiplicity, and asymptotics for the nonlinear problems involved, we state and leave several open difficult mathematical problems. Meantime, let us characterize other models involved.