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Almost periodic solution of an impulsive differential equation model of plankton allelopathy

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ABSTRACT

In this paper, we consider an impulsive differential equation model of plankton allelopathy. Sufficient conditions ensuring the existence of a unique almost periodic solution of the system are obtained, by the relation between the solutions of impulsive system and the corresponding non-impulsive system.

1. Introduction

Plankton are floating organisms of different phyla living in the pelagic regions of the sea, in freshwater lakes and in rivers. In addition to its key role at the bottom of the marine food web, phytoplankton also control the carbon recycling process which has a significant impact on regulating the climate. Because of the difficulty in measuring plankton biomass, mathematical modeling of plankton population is an important alternative method of improving our knowledge of the physical and biological processes relating to plankton ecology [1]. One of the first mathematical representations of allelopathic interactions was proposed by Maynard Smith [2]. The author considered a two species Lotka–Volterra competition model and introduced a term to take into account the effect of a toxic substance, which is released at a constant rate by one species when the other is present. The modified model takes the following form

$$\begin{aligned}\dot{x}_1(t) &= x_1(t)\left(K_1 - \alpha_1 x_1(t) - \beta_{12}x_2(t) - \gamma_1 x_1(t)x_2(t)\right),\\ \dot{x}_2(t) &= x_2(t)\left(K_2 - \alpha_2 x_2(t) - \beta_{21}x_1(t) - \gamma_2 x_1(t)x_2(t)\right),\end{aligned} \tag{1.1}$$

where the terms $b_1(t)x_1^2(t)x_2(t)$ and $b_2(t)x_1(t)x_2^2(t)$ denote the effect of toxic substances. In recent decades, many scholars have paid attention to the study of such systems and have obtained many excellent results (see [3–8] for more details).

However, the ecological system is often deeply perturbed by human exploitation activities such as planting and harvesting and so on, which makes it unsuitable to be considered continually. To obtain a more accurate description of such systems, we need to consider the impulsive differential equations. In recent years, the impulsive differential equations have been intensively investigated (see [9–20] for more detail).

In this paper, we study the following nonautonomous impulsive differential equation model of plankton allelopathy:

$$\begin{aligned}&\dot{x}_1(t) = x_1(t)[r_1(t) - a_{11}(t)x_1(t) - a_{12}(t)x_2(t) - b_1(t)x_1(t)x_2(t)],\\ &\dot{x}_2(t) = x_2(t)[r_2(t) - a_{21}(t)x_1(t) - a_{22}(t)x_2(t) - b_2(t)x_1(t)x_2(t)], \quad t \neq \tau_k,\\ &x_1(\tau_k^+) = (1 + h_{1k})x_1(\tau_k),\\ &x_2(\tau_k^+) = (1 + h_{2k})x_1(\tau_k), \quad k = 0, 1, 2, \ldots,\end{aligned} \tag{1.2}$$

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where $x_1(t), x_2(t)$ are population densities of species x_1, x_2 at time t, respectively; $r_i(t), a_{ij}(t), b_i(t), i, j = 1, 2$, are all continuous almost periodic functions which are bounded above and below by positive constants; $h_{1k}, h_{2k} > -1$ are constants and $0 = \tau_0 < \tau_1 < \tau_2 < \cdots < \tau_k < \tau_{k+1} < \cdots$, are impulse points with $\lim_{k\to+\infty} \tau_k = +\infty$.

Li, Dou and Song [13] studied the periodic case of system (1.2), and obtained some results on the permanence and extinction of the system by the comparison theorem, which cannot be used to study the existence of almost periodic solution. It is the first time for us to discuss the existence of a unique almost periodic solution of system (1.2), by the relation between the solutions of impulsive system and the corresponding non-impulsive system.

By the basic theories of impulsive differential equations in [9,10], system (1.2) has a unique solution $X(t) = X(t, X_0) \in PC([0, +\infty), R^2)$ and $PC([0, +\infty), R^2) = \{\phi : [0, +\infty) \to R^2, \phi$ is continuous for $t \neq \tau_k$. Also $\phi(\tau_k^-)$ and $\phi(\tau_k^+)$ exist, and $\phi(\tau_k^-) = \phi(\tau_k), k = 1, 2, \ldots\}$ for each initial value $x(0) = x_0 \in R^{2+}$.

The organization of this paper is as follows. In Section 2, we present some notations and lemmas. In Section 3, we study the existence of a unique almost periodic solution of system (1.2).

2. Preliminaries

Now, let us state the following definitions and lemmas, which will be useful in proving our main result.

Definition 2.1 (*[11]*)**.** The set of sequences $\{\tau_k^j = \tau_{k+j} - \tau_k\}, k, j \in Z$ is said to be uniformly almost periodic if for arbitrary $\varepsilon > 0$, there exists a relatively dense set of ε-almost periods common for any sequences.

Definition 2.2 (*[11]*)**.** The function $\varphi \in PC(R, R)$ is said to be almost periodic, if the following conditions hold:

(a) the set of sequences $\{\tau_k^j\}, k, j \in Z$ is uniformly almost periodic;
(b) for any $\varepsilon > 0$ there exists a real number $\delta > 0$ such that if the points t' and t'' belong to one and the same interval of continuity of $\varphi(t)$ and satisfy the inequality $t' - t'' < \delta$, then $\varphi(t') - \varphi(t'') < \varepsilon$;
(c) for any $\varepsilon > 0$ there exists a relatively dense set T such that if $\tau \in T$, then $\varphi(t + \tau) - \varphi(t) < \varepsilon$ for all $t \in R$, satisfying the condition $t - \tau_k > \varepsilon, k \in Z$.

For a given continuous function $g(t)$, we let g^L and g^M denote $\inf_{0\le t<+\infty} g(t)$ and $\sup_{0\le t<+\infty} g(t)$, respectively.

Consider the following system

$$\begin{aligned} \dot{y}_1(t) &= y_1(t)[r_1(t) - A_{11}(t)y_1(t) - A_{12}(t)y_2(t) - B_1(t)y_1(t)y_2(t)], \\ \dot{y}_2(t) &= y_2(t)[r_2(t) - A_{21}(t)y_1(t) - A_{22}(t)y_2(t) - B_2(t)y_1(t)y_2(t)], \end{aligned} \tag{2.1}$$

where $A_{ij}(t) = a_{ij}(t)\prod_{0<\tau_k<t}(1 + h_{jk}), B_i(t) = b_i(t)\prod_{0<\tau_k<t}(1 + h_{1k})(1 + h_{2k}), 1 \le i, j \le 2$.

Lemma 2.1. *Let $(y_1(t), y_2(t))^{\mathrm{T}}$ be any solution of system (2.1) such that $y_i(0) > 0$, then $y_i(t) > 0$ for all $t \ge 0$.*

Proof. From (2.1), we have $y_i'(t) = P_i(t)y_i(t), i = 1, 2$, where $P_i(t) = r_i(t) - A_{i1}(t)y_1(t) - A_{i2}(t)y_2(t) - B_i(t)y_1(t)y_2(t)$. Thus when $y_i(0) > 0$, we can obtain

$$y_i(t) = y_i(0)\exp\left\{\int_0^t P_i(s)\mathrm{d}s\right\} > 0.$$

This completes the proof of Lemma 2.1. □

Lemma 2.2. *For systems (1.2) and (2.1), the following results hold:*

(1) *if $(y_1(t), y_2(t))^{\mathrm{T}}$ is a solution of (2.1), then $(x_1(t), x_2(t))^{\mathrm{T}} = \left(\prod_{0<\tau_k<t}(1+h_{1k})y_1(t), \prod_{0<\tau_k<t}(1+h_{2k})y_2(t)\right)^{\mathrm{T}}$ is a solution of (1.2);*

(2) *if $(x_1(t), x_2(t))^{\mathrm{T}}$ is a solution of (1.2), then $(y_1(t), y_2(t))^{\mathrm{T}} = \left(\prod_{0<\tau_k<t}(1 + h_{1k})^{-1}x_1(t), \prod_{0<\tau_k<t}(1 + h_{2k})^{-1}x_2(t)\right)^{\mathrm{T}}$ is a solution of (2.1).*

Proof. (1) Suppose that $(y_1(t), y_2(t))^{\mathrm{T}}$ is a solution of (2.1). Let $x_i(t) = \prod_{0<\tau_k<t}(1 + h_{ik})y_i(t), i = 1, 2$, then for any $t \neq \tau_k,\ k = 1, 2, \ldots$, by substituting $y_i(t) = \prod_{0<\tau_k<t}(1 + h_{ik})^{-1}x_i(t), i = 1, 2$ into system (2.1), we can easily verify that the first two equations of (1.2) hold.

For $t = \tau_k,\ k = 1, 2, \ldots$, we have

$$\begin{aligned} x_i(\tau_k^+) &= \lim_{t\to\tau_k^+}\prod_{0<\tau_k<t}(1 + h_{ik})y_i(t) = \prod_{0<\tau_j\le\tau_k}(1 + h_{ij})y_i(\tau_k) \\ &= (1 + h_{ik})\prod_{0<\tau_j<\tau_k}(1 + h_{ij})y_i(\tau_k) = (1 + h_{ik})x_i(\tau_k). \end{aligned}$$

So the last two equations of (1.2) also hold. Thus $(x_1(t), x_2(t))^{\mathrm{T}}$ is a solution of (1.2). This proves the conclusion of (1).