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Entire solutions in the Fisher-KPP equation with nonlocal dispersal

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A B S T R A C T

This paper is concerned with entire solutions of the Fisher-KPP equation with nonlocal dispersal, i.e., $u_t = J * u - u + f(u)$, which is a one-dimensional nonlocal version of the Fisher-KPP equation describing the spatial spread of a mutant in a given population and the dispersion of the genetic characters is assumed to follow a nonlocal diffusion law modeled by a convolution operator. Here the entire solutions are defined in the whole space and for all time $t \in \mathbb{R}$. A comparison principle is employed to establish the existence of entire solutions by combining two traveling wave solutions with different speeds and coming from both ends of the real axis and some spatially independent solutions. The main difficulty is that a lack of regularizing effect occurs. This is probably the first time the existence of entire solutions of reaction equations with nonlocal dispersal has been studied.

1. Introduction and main results

This paper deals with the existence of entire solutions of nonlocal evolution equations of the form

$$u_t = J * u - u + f(u), \tag{1.1}$$

where $x \in \mathbb{R}$, $t \in \mathbb{R}$, the convolution $(J * u)(x, t) = \int_{\mathbb{R}} J(x-y)u(y, t)dy$, the kernel J is a smooth function in $\mathbb{R}$ and satisfies:

(J1) $J \in C^1(\mathbb{R}), J(x) = J(-x) \geq 0$ and $\int_{\mathbb{R}} J dy = 1$;
(J2) J is compactly supported.

The nonlinearity is induced by the function f, which we assume satisfies the following condition:

(F) $f \in C^2(\mathbb{R}), f(0) = f(1) = 0, f(u) > 0$ in $(0, 1)$, and $f'(s) \leq f'(0) < 1$ for $s \in (0, 1)$.

Obviously, if f satisfies (F), then it is the Fisher-KPP nonlinearity.

Hereafter, a traveling wave solution of (1.1) refers to a pair (ϕ_c, c), where $\phi_c = \phi_c(\xi)$ is a continuous function on $\mathbb{R}$ and $c > 0$ is a constant, such that $u(x, t) := \phi_c(x - ct) = \phi_c(\xi)$ is a solution of the equation

$$\begin{cases} J * \phi_c - \phi_c + c\phi_c' + f(\phi_c) = 0, \\ \phi_c(-\infty) = 1, \qquad \phi_c(+\infty) = 0, \end{cases} \tag{1.2}$$

where $(J * \phi_c)(\xi) = \int_{\mathbb{R}} J(\xi - y)\phi_c(y)dy$. We call c the traveling wave speed and ϕ_c the profile of such a traveling wave solution. The existence of traveling wave solutions of (1.1) under the conditions (J1), (J2) and (F) was obtained by Schumacher [1]. In fact, Carr and Chmaj [2] considered the monostable case to be more general than (1.1) and proved that if J satisfies (J1) and (J2), and $f(0) = f(1) = 0, f > 0$ in $(0, 1)$ and $f'(s) \leq f'(0)$ for $s \in (0, 1)$, then there exists a unique

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traveling wave of the form $u(x,t) = \phi_c(x - ct) = \phi_c(\xi)$ for $c \geq c^*$ up to translation (where c^* is the minimal speed) such that $\phi_c(\xi)$ satisfies (1.2), $\phi_c'(\xi) < 0$ for any $\xi \in \mathbb{R}$. Furthermore, for any $c > c^*$, there exists a positive constant $\alpha(c)$ such that

$$\phi_c(\xi) = \mathrm{e}^{-\alpha(c)\xi} + o\left(\mathrm{e}^{-\alpha(c)\xi}\right) \quad \text{as } \xi \to +\infty, \tag{1.3}$$

where

$$c\alpha(c) = \int_{\mathbb{R}} J(y)\mathrm{e}^{\alpha(c)y}\mathrm{d}y + f'(0) - 1.$$

For the minimal speed c^*,

$$\phi_{c^*}(\xi) = \xi\mathrm{e}^{-\alpha(c^*)\xi} + o\left(\xi\mathrm{e}^{-\alpha(c^*)\xi}\right) \quad \text{as } \xi \to +\infty.$$

When J does not satisfy (J2), i.e., J is not compactly supported, Coville and Dupaigne [3] obtained a more general existence result. The related results can refer to Atkinson and Reuter [4], Bates et al. [5], Berestycki and Nirenberg [6,7], Berestycki et al. [8], Brown and Carr [9], Chen [10], Coville [11], Fife and Mcleod [12], Pan et al. [13,14] and Wang et al. [15–17].

It is well-known that traveling wave solutions are special examples of the so-called entire solutions that are defined in the whole space and for all time $t \in \mathbb{R}$. Just as Morita and Ninomiya [18] pointed out, the entire solution can help us for the mathematical understanding of transient dynamics, and also can be used to imply that the dynamics of two solutions can have distinct histories in the configuration, though their asymptotic profiles as $t \to +\infty$ coincide.

Recently, some new types of entire solutions are obtained for reaction-diffusion equations with (without) nonlocal delayed nonlinearity by several authors, see Chen and Guo [19], Chen et al. [20], Fukao et al. [21], Guo and Morita [22], Hamel and Nadirashvili [23,24], Li et al. [25,26], Liu et al. [27], Wang et al. [28–30] and Yagisita [31]. The first successful example is given by Hamel and Nadirashvili [23]. In the pioneering work, they considered the Fisher-KPP equation and established five-dimensional, four-dimensional and three-dimensional manifolds of entire solutions of the equation by combining two traveling wave solutions with different speeds and coming from both sides of the real axis and some spatially independent solutions. Further, they in [24] provided the existence of entire solutions in high dimensional spaces and obtained an amazingly rich class of entire solutions by using traveling wave fronts. In addition, by constructing a global invariant manifold with asymptotic stability, Yagisita [31] proved, for the bistable equation, that there exists an entire solution which behaves as two traveling wave solutions coming from both sides of the x-axis and annihilating in a finite time, and the stability and uniqueness of the entire solution were also considered. Yagisita's argument was substantially simplified by Fukao et al. [21]. Chen and Guo [19] and Guo and Morita [22] developed an unified approach based on the comparison principle to find entire solutions for both the bistable and monostable cases. Chen et al. [20] showed the existence and uniqueness of entire solutions in reaction-diffusion equations with balanced bistable nonlinearity. Here the balanced bistable nonlinearity means the wave speed $c = 0$. More recently, Li et al. [25] established the existence of entire solutions of reaction-advection-diffusion equations in infinite cylinders.

We know that equations like $u_t = J * u - u$ and variations of it, have been recently widely used to model diffusion processes, for example, in biology, dislocations dynamics, etc. See, for example, [32,9,33,34,14]. As stated in [35], if $u(x,t)$ is thought of as the density of a single population at the point x at time t, and $J(x - y)$ is thought of as the probability distribution of jumping from location y to location x, then $(J * u)(x,t) = \int_{\mathbb{R}} J(x-y)u(y,t)\mathrm{d}y$ is the rate at which individuals are arriving at position x from all other places and $-u(x,t) = -\int_{\mathbb{R}} J(y-x)u(x,t)\mathrm{d}y$ is the rate at which they are leaving location x to travel to all other sites. This consideration, in the absence of external or internal sources, leads immediately to the fact that the density u satisfies the equation $u_t = J * u - u$.

In our case, see the equation in (1.1), we have a diffusion operator $J * u - u$ and a nonlinear reaction term given by $f(u)$. It is known from [35] that if J is nonnegative and compactly supported, then the equation $u_t = J * u - u$ shares many properties with the classical heat equation $u_t = \Delta u$, such as: bounded stationary solutions are constant, a maximum principle holds for both of them and perturbations propagate with infinite speed. Thus, (1.1) can be regarded as a nonlocal analogous of the usual reaction-diffusion equation

$$u_t = u_{xx} + f(u). \tag{1.4}$$

Motivated by the results due to Hamel and Nadirashvili [23], we expect naturally to build some new entire solutions of (1.1). Resolving this issue represents a main contribution of our current study.

Before stating the main results of this paper, we first recall some known results of entire solutions of the following equation

$$u_t = D[u(x+1,t) + u(x-1,t) - 2u(x,t)] + f(u), \tag{1.5}$$

which were studied by Guo [36] and Guo and Morita [22]. Especially, Guo and Morita [22, Theorem 1.4, Section 4] considered a general quasilinear discrete diffusion equation with Fisher-KPP nonlinearity and established some existence results of entire solutions. Guo [36] considered the (1.5) with bistable nonlinearity and obtained results similar to those of [18].

In this paper, we construct some new types of entire solutions of (1.1) by combining two traveling wave solutions with different speeds and coming from both sides of the real axis and some spatially independent solutions. Each of those traveling