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Four positive periodic solutions to a periodic Lotka–Volterra predatory–prey system with harvesting terms

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ABSTRACT

By using Mawhin's continuation theorem of coincidence degree theory, we establish the existence of four positive periodic solutions for two species periodic Lotka–Volterra predatory–prey system with harvesting terms. An example is given to illustrate the effectiveness of our results.

1. Introduction

Two species Lotaka–Volterra predatory–prey model with harvesting terms is formulated by [1,2]

$$\begin{cases} \dot{x} = x(t)(a_1 - b_1 x(t) - c_1 y(t)) - h_1, \\ \dot{y} = y(t)(a_2 - b_2 y(t) + c_2 x(t)) - h_2, \end{cases}$$

where $x(t)$ and $y(t)$ denote the densities of the prey and the predator, respectively; a_i and b_i $(i = 1, 2)$ are all positive constants and denote the intrinsic growth rates and the intra-specific competition rates, respectively; $c_1 > 0$ is the predation rate of the predator and $c_2 > 0$ represents the conversion rate at which the ingested prey in excess of what is needed for maintenance is translated into the predator population increase; h_i $(i = 1, 2)$ is the ith species harvesting terms standing for the harvests. Since realistic models require the inclusion of the effect of changing environment, this motivates us to consider the following nonautonomous model:

$$\begin{cases} \dot{x} = x(t)(a_1(t) - b_1(t)x(t) - c_1(t)y(t)) - h_1(t), \\ \dot{y} = y(t)(a_2(t) - b_2(t)y(t) + c_2(t)x(t)) - h_2(t). \end{cases} \tag{1.1}$$

In addition, the effects of a periodically varying environment are important for evolutionary theory as the selective forces on systems in a fluctuating environment differ from those in a stable environment. Therefore, the assumptions of periodicity of the parameters are a way of incorporating the periodicity of the environment (e.g., seasonal effects of weather, food supplies, mating habits, etc.), which leads us to assume that $a_i(t)$, $b_i(t)$, $c_i(t)$ and $h_i(t)$ $(i, j = 1, 2)$ are all positive continuous ω-periodic functions.

Since a very basic and important problem in the study of a population growth model with a periodic environment is the global existence and stability of a positive periodic solution, which plays a similar role as a globally stable equilibrium

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does in an autonomous model, also, on the existence of positive periodic solutions to system (1.1), few results are found in literatures. This motivates us to investigate the existence of a positive periodic or multiple positive periodic solutions for system (1.1). In fact, it is more likely for some biological species to take on multiple periodic change regulations and have multiple local stable periodic phenomena. Therefore, it is essential for us to investigate the existence of multiple positive periodic solutions for population models. Our main purpose of this paper is by using Mawhin's continuation theorem of coincidence degree theory [3] to establish the existence of four positive periodic solutions for system (1.1). For the work concerning the multiple existence of periodic solutions of periodic population models which was done using coincidence degree theory, we refer to [4–6].

The organization of the rest of this paper is as follows. In Section 2, by employing the continuation theorem of coincidence degree theory, we establish the existence of four positive periodic solutions of system (1.1). In Section 3, an example is given to illustrate the effectiveness of our results.

2. Existence of four positive periodic solutions

In this section, by using Mawhin's continuation theorem, we shall show the existence of positive periodic solutions of (1.1). To do so, we need to make some preparations.

Let X and Z be real normed vector spaces. Let $L : \operatorname{Dom} L \subset X \to Z$ be a linear mapping and $N : X \times [0, 1] \to Z$ be a continuous mapping. The mapping L will be called a Fredholm mapping of index zero if $\dim \operatorname{Ker} L = \operatorname{codim} \operatorname{Im} L < \infty$ and $\operatorname{Im} L$ is closed in Z. If L is a Fredholm mapping of index zero, then there exist continuous projectors $P : X \to X$ and $Q : Z \to Z$ such that $\operatorname{Im} P = \operatorname{Ker} L$ and $\operatorname{Ker} Q = \operatorname{Im} L = \operatorname{Im}(I - Q)$, and $X = \operatorname{Ker} L \bigoplus \operatorname{Ker} P$ and $Z = \operatorname{Im} L \bigoplus \operatorname{Im} Q$. It follows that $L|_{\operatorname{Dom} L\cap \operatorname{Ker} P} : (I - P)X \to \operatorname{Im} L$ is invertible and its inverse is denoted by K_P. If Ω is a bounded open subset of X, the mapping N is called L-compact on $\bar{\Omega} \times [0, 1]$, and if $QN(\bar{\Omega} \times [0, 1])$ is bounded and $K_P(I - Q)N : \bar{\Omega} \times [0, 1] \to X$ is compact. Because $\operatorname{Im} Q$ is isomorphic to $\operatorname{Ker} L$, there exists an isomorphism $J : \operatorname{Im} Q \to \operatorname{Ker} L$.

The Mawhin's continuous theorem [3, p. 40] is given as follows.

Lemma 2.1 (*[3]*). *Let L be a Fredholm mapping of index zero and let N be L-compact on $\bar{\Omega} \times [0, 1]$. Assume*

(a) *for each $\lambda \in (0, 1)$, every solution x of $Lx = \lambda N(x, \lambda)$ is such that $x \notin \partial\Omega \cap \operatorname{Dom} L$;*
(b) *$QN(x, 0)x \neq 0$ for each $x \in \partial\Omega \cap \operatorname{Ker} L$;*
(c) $\deg(JQN(x, 0), \Omega \cap \operatorname{Ker} L, 0) \neq 0$.

Then, $Lx = N(x, 1)$ has at least one solution in $\overline{\Omega} \cap \operatorname{Dom} L$.

For the sake of convenience, we denote $f^l = \min_{t\in[0,\omega]} f(t), f^M = \max_{t\in[0,\omega]} f(t), \bar{f} = \frac{1}{\omega}\int_0^\omega f(t)\mathrm{d}t$; here $f(t)$ is a continuous ω-periodic function.

Throughout this paper, we need the following assumptions.

(H_1) $a_1^l - c_1^M H_1 > 2\sqrt{b_1^M h_1^M}$;
(H_2) $a_2^l > 2\sqrt{b_2^M h_2^M}$;
(H_3) $c_2^M l_1^+ > \sqrt{(a_2^l)^2 - 4b_2^M h_2^M}$.

For simplicity, we also introduce some positive numbers as follows:

$$l_1^\pm = \frac{a_1^M \pm \sqrt{(a_1^M)^2 - 4b_1^l h_1^l}}{2b_1^l}, \qquad A^\pm = \frac{(a_1^l - c_1^M H_1) \pm \sqrt{(a_1^l - c_1^M H_1)^2 - 4b_1^M h_1^M}}{2b_1^M},$$

$$l_2^\pm = \frac{a_2^l \pm \sqrt{(a_2^l)^2 - 4b_2^M h_2^M}}{2b_2^M}, \qquad H_1 = \frac{a_2^M + c_2^M l_1^+}{b_2^l}, \qquad H_2 = \frac{h_2^l}{a_2^M + c_2^M l_1^+}.$$

Lemma 2.2. *Let $x > 0, y > 0, z > 0$ and $x > 2\sqrt{yz}$, for the functions $f(x, y, z) = \frac{x+\sqrt{x^2-4yz}}{2z}$ and $g(x, y, z) = \frac{x-\sqrt{x^2-4yz}}{2z}$, the following assertions hold.*

(1) *$f(x, y, z)$ and $g(x, y, z)$ are monotonically increasing and monotonically decreasing on the variable $x \in (0, \infty)$, respectively.*
(2) *$f(x, y, z)$ and $g(x, y, z)$ are monotonically decreasing and monotonically increasing on the variable $y \in (0, \infty)$, respectively.*
(3) *$f(x, y, z)$ and $g(x, y, z)$ are monotonically decreasing and monotonically increasing on the variable $z \in (0, \infty)$, respectively.*

Proof. In fact, for all $x > 0, y > 0, z > 0$, we have

$$\frac{\partial f}{\partial x} = \frac{x + \sqrt{x^2 - 4yz}}{2z\sqrt{x^2 - 4yz}} > 0, \qquad \frac{\partial g}{\partial x} = \frac{\sqrt{x^2 - 4yz} - x}{2z\sqrt{x^2 - 4yz}} < 0, \qquad \frac{\partial f}{\partial y} = \frac{-1}{\sqrt{x^2 - 4yz}} < 0,$$

$$\frac{\partial g}{\partial y} = \frac{1}{\sqrt{x^2 - 4yz}} > 0, \qquad \frac{\partial f}{\partial z} = \frac{-x(x + \sqrt{x^2 - 4yz})}{2z^2\sqrt{x^2 - 4yz}} < 0, \qquad \frac{\partial g}{\partial z} = \frac{x(x - \sqrt{x^2 - 4yz})}{2z^2\sqrt{x^2 - 4yz}} > 0.$$