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Dynamics of a new Lorenz-like chaotic system

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ABSTRACT

The present work is devoted to giving new insights into a new Lorenz-like chaotic system. The local dynamical entities, such as the number of equilibria, the stability of the hyperbolic equilibria and the stability of the non-hyperbolic equilibrium obtained by using the center manifold theorem, the pitchfork bifurcation and the degenerate pitchfork bifurcation, Hopf bifurcations and the local manifold character, are all analyzed when the parameters are varied in the space of parameters. The existence of homoclinic and heteroclinic orbits of the system is also rigorously studied. More exactly, for $b \geq 2a > 0$ and $c > 0$, we prove that the system has no homoclinic orbit but has two and only two heteroclinic orbits.

1. Introduction

There has been considerable research as regards chaos since the first remarkable chaotic model, i.e. the Lorenz system [1,2], was presented. Chaos, though it is extremely and intrinsically complex and disordered, can also be beneficial to many real-world applications, including fluid mixing, secure communication, heating and some other scientific and engineering applications. In contrast, for some circumstances, chaos is undesirable and should be weakened or eliminated. To sum up, chaos needs a thorough understanding of its physical essence, thereby making it more useful. During recent years, reports about chaos have concentrated on not only proposing new and interesting chaotic systems (the Chen system [3], the Lü system [4], the Lorenz system family [5], the conjugate Lorenz-type system [6], the Yang system [7] and others) or studying chaos control and chaos synchronization [8], but also theoretically analyzing their local and global characteristics which are essential for an understanding of what is meant by chaos. For example, there have been some detailed bifurcation analyses concerning the Chen system [9–13], the Lü system [14,15] and others [16] such as pitchfork bifurcation, Hopf bifurcation, homoclinic bifurcation, tangent bifurcation and some other complex dynamical behaviors, which all shows that these systems have rich nonlinear dynamics and are of significance in practical application. There are also some other papers devoted to the mathematical analysis of some other chaotic systems.

It is very important to note that almost all such 3D autonomous chaotic systems have three particular equilibria: one saddle and two unstable saddle-foci (for example, the Lorenz system [2], the Chen system [3], the Lü system [4], the conjugate Lorenz-type system [6] etc). The other 3D chaotic systems, such as the original Rössler system [17] and the diffusionless Lorenz system (DLS) [18], have two unstable saddle-foci.

From the point of view of the potential applications, systems with sensitivity to the initial conditions can be used in secure communications [19,20]. Among the pioneering papers which proposed using chaotic systems in communications are the papers of Pecora and Carroll [21,22]. Consequently, an appropriate chaotic system can be chosen from a catalogue of chaotic systems to optimize some desirable factors, an idea suggested in [20].

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These same ideas led us to study a new Lorenz-like chaotic system, which is described via the following three-dimensional smooth quadratic autonomous system [7]:

$$\begin{cases} \dot{x} = a(y - x) \\ \dot{y} = cx - xz \\ \dot{z} = -bz + xy, \end{cases} \tag{1.1}$$

where $(a, b, c) \in (0, \infty) \times (-\infty, \infty) \times (-\infty, \infty)$ is the parameter vector. When $(a, b, c) = (10, 8/3, 16)$ and $(a, b, c) = (35, 3, 35)$, system (1.1) has the following Lyapunov exponents: $\lambda_{LE_1} = 0.57018$, $\lambda_{LE_2} = 0.0000$, $\lambda_{LE_3} = -13.237$ and $\lambda_{LE_1} = 1.0742$, $\lambda_{LE_2} = 0.0000$, $\lambda_{LE_3} = -39.074$, respectively. In particular, when $(a, b, c) = (35, 3, 35)$, the system has very different fixed points: one saddle and two stable node-foci. It is immediately clear that the system is topologically nonequivalent to the original Lorenz and the other Lorenz-like systems. Moreover, in the sense defined by Vaněček and Čelikovský [23], system (1.1) also connects the original Lorenz system and the original Chen system and represents a transition from one to the other. Therefore, what is interesting is to further find out what kind of new dynamics this system has.

Homoclinic orbits and heteroclinic orbits are important concepts in the study of the bifurcation of vector fields and chaos. Many chaotic behaviors of a complex system are related to the existence or nonexistence of these kinds of orbits in the system. The study of homoclinic orbits and heteroclinic orbits for a chaotic system is an important and yet difficult task in nonlinear dynamical systems theory. Recently, the study of some chaotic systems has revealed the fundamental factor that explains how the difficulties came into being [24,25]. Recall that a homoclinic orbit is a trajectory that is doubly asymptotic to an equilibrium point, or is a closed orbit asymptotic to itself. A heteroclinic orbit is a trajectory that connects an equilibrium point or a closed orbit to another equilibrium point or another closed orbit, respectively.

In this paper we give new insights into a new Lorenz-like chaotic system (1.1). Aiming to contribute to the understanding of the dynamics of this new chaotic system, we present some new results of an analytical bifurcation analysis of its solutions when the parameters are varied, and analyze the existence of homoclinic orbits and heteroclinic orbits.

The paper is organized as follows. In Section 2 we discuss the local dynamics in the space of parameters — such as the number of equilibria, the stability of hyperbolic equilibria and the stability of the non-hyperbolic equilibrium — by using the center manifold theorem, the pitchfork bifurcation and the degenerate pitchfork bifurcation, Hopf bifurcations and the local manifold character. In Section 3 the existence of homoclinic orbits and heteroclinic orbits is also rigorously studied analytically using techniques from the literature [24]. Finally, conclusions are drawn in Section 4.

2. Local bifurcation

Consider system (1.1). The equilibria can be found by solving the following algebraic equations:

$$a(y - x) = 0, \qquad cx - xz = 0, \qquad xy - bz = 0.$$

Obviously, when $bc < 0$, $S_0 = (0, 0, 0)$ is a unique equilibrium. In addition, under the condition $bc > 0$, the system also has two nonzero equilibria, $S_\pm = (\pm\sqrt{bc}, \pm\sqrt{bc}, c)$. If $b = 0$, the system has a non-isolated equilibrium $(0, 0, z)$ $(z \in \mathbb{R})$. If $c = 0$, the system has a unique equilibrium $S_0 = (0, 0, 0)$.

The Jacobian matrix of system (1.1) at the origin S_0 is

$$A = \begin{pmatrix} -a & a & 0 \\ c & 0 & 0 \\ 0 & 0 & -b \end{pmatrix}.$$

The characteristic equation of A is

$$(\lambda^2 + a\lambda - ac)(\lambda + b) = 0. \tag{2.1}$$

Suppose that $c = 0$. Eq. (2.1) has three roots $\lambda_1 = 0$, $\lambda_2 = -a$ and $\lambda_3 = -b$, with the corresponding eigenvectors being $(1, 1, 0)$, $(1, 0, 0)$ and $(0, 0, 1)$. Hence, S_0 is non-hyperbolic, and if $b < 0$ then $\lambda_3 = -b > 0$. Thus, the origin S_0 is unstable. Next, we will investigate the stability of S_0 for $b > 0$ by using the center manifold theorem.

Proposition 2.1. *Assume that $c = 0$ and $a > 0$. If $b > 0$ then the non-hyperbolic equilibrium S_0 is asymptotically stable.*

Proof. From the above discussion, the origin S_0 is non-hyperbolic with three eigenvalues 0, $-a$ and $-b$. Next, we will investigate the stability at S_0 by using the center manifold theorem.

Let us have the transformation

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}.$$

Transform system (1.1) into

$$\begin{pmatrix} \dot{x}_1 \\ \dot{y}_1 \\ \dot{z}_1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & -b \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + \begin{pmatrix} -(x_1 + y_1)z_1 \\ (x_1 + y_1)z_1 \\ (x_1 + y_1)x_1 \end{pmatrix}. \tag{2.2}$$