Nonlinear Analysis: Real World Applications 11 (2010) 3139–3154

Contents lists available at ScienceDirect

Nonlinear Analysis: Real World Applications

journal homepage: www.elsevier.com/locate/nonrwa

Analysis and short-time extrapolation of stock market indexes through projection onto discrete wavelet subspaces

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ARTICLE INFO

Article history:
Received 20 May 2009
Accepted 4 November 2009

Keywords:
Multi-resolution analysis
Wavelet decomposition
Scale-limited extrapolation
Empirical finance

ABSTRACT

We consider the problem of short-time extrapolation of blue chips' stocks indexes in the context of wavelet subspaces following the theory proposed by X.-G. Xia and co-workers in a series of articles [10–13]. The idea is first to approximate the oscillations of the corresponding stock index at some scale by means of the scaling function which is part of a given multi-resolution analysis of $L^2(\mathbb{R})$. Then, since oscillations at a finer scale are discarded, it becomes possible to extend such a signal up to a certain time in the future; the finer the approximation, the shorter this extrapolation interval. At the numerical level, a so-called Generalized Gerchberg–Papoulis (GGP) algorithm is set up which is shown to converge toward the minimum L^2 norm solution of the extrapolation problem. When it comes to implementation, an acceleration by means of a Conjugate Gradient (CG) routine is necessary in order to obtain quickly a satisfying accuracy. Several examples are investigated with different international stock market indexes.

1. Introduction

Extrapolation is one of the fundamental problems arising in Signal Processing; during the last decades, a lot of work has been devoted to it, in the particular context of band-limited signals, see e.g. [1–7]. The theoretical basis for band-limited extrapolation has to do with the classical Paley–Wiener theorem which states that a function belonging to $L^2(\mathbb{R})$ whose Fourier transform has compact support can be extended to the complex plane as an entire function of exponential type. Hence, its knowledge restricted to any connected open set still allows for a reconstruction up to any arbitrarily big domain by relying onto classical analytic functions theory. Numerically, things are more involved, especially because this analytic extension turns out to be an ill-conditioned problem which is quite sensitive to noise and truncation errors; in particular, methods based on Taylor series have been completely given up for Fast Fourier Transform (FFT) routines, see again [1,2,4,5,7].

In this article, we follow a similar methodology to produce reliable extrapolations of liquid stock market indexes relying on the so-called *scale-limited* approximation. Roughly speaking, in the case of band-limited extrapolation, one assumes that the signal under consideration can be approximated by a function of compact support in the Fourier space (hence, having only finite frequency oscillations) whereas in the present context, one considers that the signal can be well represented by its projection onto a subspace of $L^2(\mathbb{R})$ containing functions with rather "thick" oscillations. Such subspaces can be found by considering what is now usually called a *Multi-Resolution Analysis* (MRA)of $L^2(\mathbb{R})$ (see Definition 1): they are spanned by bases of shift-invariant *scaling functions* with various smoothness properties [8]. Hence, this work applies primarily to indexes which are dominated by their low-frequency approximation and for which fine scale features can be considered as a detail not being of great importance.

This is not a very restrictive assumption, since the composition of most of the modern stock indexes is regularly updated to keep their variations under control even when some share's prices may have erratic behavior. For instance, on June 18,

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1468-1218/$ – see front matter
doi:10.1016/j.nonrwa.2009.11.009

2007, AGF and Thomson have been removed from the French index CAC 40; they have been replaced by Air France-KLM and Unibail. Similarly, the Dow Jones Industrials 30 Index will see its composition modified as General Motors should leave because its capitalization dropped too strongly during March 2009. An estimate of the volatility of an investment can be given by a statistical measure known as the standard deviation of the return rate. Standard deviation can be thought as being synonymous with volatility in such a context. An S&P 500 index fund has a standard deviation of about 15%; a standard deviation of zero would mean a return rate that never varies, like a bank account paying compound interest at a guaranteed rate. Exchange rates on FOREX markets seem to display quite an opposite behavior and the methods presented in this article do not seem to apply (see e.g. [9] for some empirical differences between stock and FX markets, especially the gain/loss asymmetry).

Recently, a whole theory has been proposed in a series of articles [10–13] (see also [14]) to extend the well-known band-limited extrapolation theory to a more flexible framework of scale-limited spaces, which have the nice property to be able to handle finite-duration signals. The roadmap really follows classical Fourier extrapolation and proceeds through linear integral operator theory (see e.g. [15] for a general reference): this is recalled in Section 2.1. However, one big difference is that from a compact set of observations, one cannot theoretically reconstruct the signal on the whole real line (except in the special case, where the MRA involves bases of analytic functions), see Theorem 3. So one can merely hope to reconstruct a small part of the missing signal on both sides of the measures interval; hence, the "short-time extrapolation" in the title. The size of this "small part" is related to the smoothness of the analyzing wavelets and the scale at which one decides to work (and this scale itself is related to the volatility of the index).

In Section 2.2, we present the standard algorithms to produce these extrapolations: they originate from the now classical Gerchberg–Papoulis algorithm [1,2] widely used for band-limited extrapolation and whose convergence has been studied in various articles, see [16,4,5,3] and references therein. In the present case, this routine converges toward the so-called *Minimum-norm Solution* of the extrapolation problem. Due to its poor practical convergence properties, an acceleration by the *Conjugate Gradient* (CG) algorithm is generally needed; this is explained in [4,7] in the case of band-limited extrapolation, and in [12] for scale-limited extrapolation. We do not completely follow this article in Section 2.3, as we propose an easier way to produce a CG acceleration.

Our specific application is considered in Section 3. First, in Section 3.1, we explain that because of the features of the market indexes, particular wavelet bases are needed. Namely, since these indexes are strongly non-periodic, one cannot consider usual wavelet bases defined on the whole real line as they process finite-duration signals either by *zero-padding* or by artificial periodization. Both techniques will lead to the appearance of strong discontinuities at the edge of the observation interval, which will be reflected in the wavelet coefficients at very fine scales. Hence, such bases will yield poor representations at medium scales, especially in the vicinity of the borders. A solution is to use the *wavelet bases on a bounded interval*: these bases have been first proposed by Meyer [17], then by Cohen, Daubechies, Jawerth and Vial [18,19]; see also *e.g.* [20,8,21,22]. Then, in Section 3.2, we explain through an example that the results can be of quite poor quality when scale-limited extrapolation is applied to individual share prices displaying locally strong variations on a fine scale. Smoothness and weak volatility are, therefore, requested to produce an acceptable result.

In Section 4, we present some numerical results on real-life data. First, in Section 4.1, a quiet bull market on the French CAC 40 is considered: this gives the best conditions in which the scale-limited extrapolation can be used because the market follows a stable trend and the volatility is low. Then, in Section 4.2, we look at the real-estate EPRA Eurozone index as a prototype of a moderately volatile stock index in a bear market; the quality of the extrapolation is lower, but there are signs (like the size of some wavelet coefficients) which give a hint about the difficulty in treating such a situation. Finally, we considered what happened on the Hang-Seng Chinese index during the first days of October 2008, when it lost around 25% of its value in less than 10 days; obviously, extrapolation routines cannot give reliable information in such a case, but the analysis of what is left behind after projecting the index onto a scale-limited subspace of $L^2(\mathbb{R})$ can somewhat reveal itself as a warning sign its high volatility (which makes violent corrections plausible in a difficult economic period). In Appendix, we briefly present the results on scale-limited extrapolation of assets log-returns (see [9]).

Let us recall finally that other works previously addressed the wavelet techniques for stock market data: see the review article [23] and the book [24]. Concerning this book, the authors work primarily with the discontinuous Haar wavelet which in general forbids to use any of these scale-limited extrapolation techniques. Other forecasting algorithms are proposed in [25–29]. A book is devoted to wavelet techniques for time series analysis, see [30]. Some results of empirical finance are reviewed in [9].

2. From band-limited to scale-limited extrapolation

In this section, we aim at reviewing some of the results from [10–13] which will be useful in the context of processing stock market indexes. Let us begin by introducing the concept of MRA: (see e.g. [8] for details)

Definition 1. A sequence of nested subspaces V_j is called a MRA of $L^2(\mathbb{R})$ if: $\{0\} \subset \cdots \subset V_{-1} \subset V_0 \subset V_1 \subset \cdots \subset L^2(\mathbb{R})$. Moreover, the following properties must hold:

- for all $f \in L^2(\mathbb{R})$, $\|\mathcal{P}_{V_j} f - f\|_{L^2} \to 0$ as $j \to +\infty$ also, $\mathcal{P}_{V_j} f \to 0$ as $j \to -\infty$.
- if $f(t) \in V_j$, then $f(t/2) \in V_{j-1}$ and for all $k \in \mathbb{Z}$, $f(t - 2^j k) \in V_j$.
- there exists a shift-invariant orthonormal base of V_0 given by the scaling function $\phi_n(t) = \phi(t - n)$ for $n \in \mathbb{Z}$.