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Existence of solutions and stability analysis for a Darcy flow with extraction☆

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ABSTRACT

In [C.J. van Duijn, G. Galiano, M.A. Peletier, A diffusion-convection problem with drainage arising in the ecology of mangroves, Interfaces Free Bound. 3 (2001) 15–44], a one-dimensional model describing the vertical movement of water and salt in a porous medium in which a continuous extraction of fresh water takes place was studied. Among other results, it was shown that for some range of parameter values, a heavier layer of water is formed above a lighter one in the transient state with a unique stable steady state. In this paper, we study the N-dimensional spatial model, for which Darcy's law must be introduced in the flow description. We prove the existence and uniqueness of weak solutions to the time evolution problem and perform a heuristic stability analysis in two ways: analytically, for a related problem, to find an approximation of the bifurcation curve in terms of the Rayleigh number, and numerically, to show the formation of instabilities in the original problem and their influence on the speed of convergence towards the stable steady state.

1. Introduction

Consider a water saturated bounded porous medium with horizontal upper and lower boundaries containing a solute, and suppose that an extracting mechanism within the upper part of the medium produces an upward flow of fresh water out through the upper boundary while keeping most of the solute content within the medium. If the fresh water extraction is strong enough, then a solute high concentration layer is created in the extraction region on the top of a lower concentration region and, therefore, gravitationally driven instabilities are expected to arise. This is a well known phenomenon observed in the ecology of mangroves, see [15,4]. In Appendix we deduce the following mathematical model. Let $u \in [0,1]$ be the solute concentration, $\mathbf{q}$ the water flow discharge and p the pressure, and consider the domain $Q_T = \Omega \times (0,T)$ for $T > 0$ and $\Omega = B \times (0,1)$, with $B \subset \mathbb{R}^{N-1}$ bounded. Find $u, p : \bar{Q}_T \to \mathbb{R}$ and $\mathbf{q} : \bar{Q}_T \to \mathbb{R}^N$ such that

$$u_t + \operatorname{div}(Ru\mathbf{q} - \nabla u) = 0, \tag{1}$$

$$\operatorname{div}\mathbf{q} + mf(\cdot, u) = 0, \tag{2}$$

$$\mathbf{q} + \nabla p - u\mathbf{e}_z = 0, \tag{3}$$

in Q_T. Positive parameters R and m stand for the Rayleigh and the *extraction* numbers of the physical system, see (82) and (83). The vector $\mathbf{e}_z$ is the canonical vertical vector pointing downwards. In (2), the extraction function $f : \bar{B} \times [0,1] \times [0,1] \to \mathbb{R}_+$

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is usually assumed to have the form

$$f(\mathbf{x}, z, \sigma) := s(z)(1-\sigma)_+^r, \tag{4}$$

with $r > 0$ and s describing the localization of the extraction region, given by (for $d \in (0, 1)$)

$$s(z) := \begin{cases} 1 & \text{if } z \in [0, d], \\ 0 & \text{if } z \in (d, 1]. \end{cases} \tag{5}$$

The spatial boundary is decomposed as $\partial\Omega = \Gamma_D \cup \Gamma_N$, with $\Gamma_D = B \times \{0\}$ and $\Gamma_N = (B \times \{1\}) \cup (\partial B \times (0, 1))$. The following boundary conditions are prescribed

$$u = u_D, \quad p = 0 \quad \text{on } \Gamma_D \times (0, T), \tag{6}$$

$$\nabla u \cdot \mathbf{n} = \mathbf{q} \cdot \mathbf{n} = 0 \quad \text{on } \Gamma_N \times (0, T). \tag{7}$$

A non-negative initial distribution, u_0, is considered to close the problem

$$u(\cdot, 0) = u_0 \quad \text{in } \Omega. \tag{8}$$

In this article we are interested in two questions: first is proving the existence and uniqueness of solutions of problem (1)–(3) and (6)–(8), which we shall refer to as *Problem P*. Second is a stability issue. In [8], in the context of one dimensional spatial variable (depth) it is proven that if the exponent r in function f, see (4), is smaller than one then the solute concentration may reach the threshold value $u = 1$ in finite time in some subset of $(0, d)$, while $u < 1$ below that layer for all $T < \infty$. This is clearly an instable situation and it is therefore expectable to observe gravitational instabilities when perturbations of the one-dimensional profile are considered in the N-dimensional setting. Our aim is to provide a range of values for the bifurcation parameter, R, for which these instabilities appear.

Existence of solutions for Darcy flows is already established for a large variety of physical situations which translate into different mathematical models, among which the most treated in the literature are the porous medium equation, the black oil system (two-phase filtration problem) and the dam problem see, for instance, [2,11,7] and the references therein. Many of these problems neglect the gravity effects expressed by the term $u\mathbf{e}_z$ in the Darcy's equation (3) and set the problem only in terms of the concentration and the pressure. Similarly, system (1)–(3) may be reduced to equations

$$u_t - \operatorname{div}(Ru(\nabla p - u\mathbf{e}_z) + \nabla u) = 0, \tag{9}$$

$$-\Delta p + \frac{\partial u}{\partial z} + mf(\cdot, u) = 0. \tag{10}$$

Note that when gravity effects, expressed by the term $\partial u/\partial z$, may be neglected then the resulting problem lies in the general setting studied, for instance, in [1]. However, if gravity effects have to be taken into account then this reduction is not appropriate. The reason is that the key role of $\operatorname{div} \mathbf{q} \in L^\infty$ is hidden in formulation (9) and (10), and a usual fixed point technique for proving existence of solutions by compactness arguments lead to the consideration of Sobolev spaces with somehow rare dimension-dependent exponents. On the contrary, direct consideration of system (1)–(3) leads to a simpler proof of existence of solutions by uncoupling the original system in two sets of equations with independent physical meaning: concentration evolution with prescribed convection, on the one hand, and flow-pressure balance with prescribed concentration, on the other. In addition, the numerical scheme corresponding to this approach is more efficient, see [5,6,13].

The stability properties of equations (1)–(3) has also received attention for a variety of data, and phenomena like cellular convection or fingering have been proven to arise when the bifurcation parameter, R, is large enough. For instance, the steady state one-dimensional solution of the model problem $L = \infty, f = 0, \mathbf{q} \cdot \mathbf{n} = 0$ on $\partial\Omega \times (0, T)$ and $u = 1$ on $\Gamma_D \times (0, T)$ is known to be unstable for values $R > 4\pi^2$, see for instance [16]. Other interesting models related to ours which also lead to gravitational instabilities are the salt lake formation by evaporation ($q \cdot \mathbf{n} = -const.$ on $\Gamma_D \times (0, T)$), see [9], or the peat moss formation ($f = 0$ and the temperature $u = u_D(t)$ on Γ_D), see [17]. The common feature of these models is the existence of an unstable ground state, i.e. a steady one-dimensional solution which may be gravitationally unstable. Analysis of the perturbation equations (linearized or not) and the study of a maximization problem for the bifurcation parameter is the usual approach for finding the threshold value of R above which instabilities occur.

For Problem P, due to the non-zero extraction term f and the non-flow boundary condition $\mathbf{q} \cdot \mathbf{n} = 0$ on $\Gamma_N \times (0, T)$, the one-dimensional steady state solution is stable, see [8]. Therefore, what we mean by instabilities associated with solutions of Problem P is slightly different from the common use. Instabilities in solutions of Problem P appear, if they do, only in the transient state, when there exists the possibility of the formation of a layer of heavier fluid above a layer of lighter fluid. However, when $t \to \infty$ these instabilities diminish in size and do disappear in infinite time. A rigorous mathematical analysis of this phenomenon is out of our scope but its physical interest, which resides in the shortening of the time rate at which solutions to the evolution problem approach to the steady state, induce us to study a related problem which may be treated rigorously, and to demonstrate by means of numeric simulations that the behavior of solutions to this related problem and to the original problem are similar, at least in the selected parameters range.

Finally, let us say some words about the uniqueness of solutions. In [8], for the one-dimensional model, we proved a partial result for a general non-Lipschitz continuous extraction term, f, which, apart from requiring some technical conditions on the solution to hold true, employed techniques which can not be extended to the multi-dimensional case.