Nonlinear Analysis: Real World Applications 10 (2009) 2133–2138

Contents lists available at ScienceDirect

Nonlinear Analysis: Real World Applications

journal homepage: www.elsevier.com/locate/nonrwa

A note on decay of potential vortex in an Oldroyd-B fluid through a porous space

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ARTICLE INFO

Article history:
Received 18 February 2008
Accepted 25 March 2008

Keywords:
Modified Darcy's law
Exact solutions
Hankel transform
Tangential tension

ABSTRACT

The problem of decay of a potential vortex in an Oldroyd-B fluid filling the porous space is studied. The flow problem is first modeled and then solved by employing the Hankel transform. Analytical expressions of the velocity field and the associated tangential tension are developed. The well known solutions for a Newtonian fluid as well as those corresponding to a Maxwell fluid and a second grade one, appear as limiting cases of the present solutions. Finally, some graphical results describing the influence of porous space parameters are sketched and interpreted.

1. Introduction

It is well known that the Navier–Stokes equations can describe the flow behaviors of viscous fluids. Such equations are unable to analyze the rheological characteristics of non-Newtonian fluids. Because of practical applications in industry and technology the flows of non-Newtonian fluids have been greatly acknowledged in recent times by the physicists, engineers, modelers, computer scientists and mathematicians as well. The governing equations of non-Newtonian fluids are much more complicated, higher order [1,2] and pose challenges in coping with the non-linearity of the involved expressions, field coupling and complex boundary conditions. In view of all these challenges many workers [3–10] have recently engaged in developing analytical solutions of the problem involving non-Newtonian fluids through various aspects. It is also a recognized fact that all non-Newtonian fluids cannot be described by a single constitutive equation. Therefore various constitutive equations of non-Newtonian fluids are suggested in the literature. Motivated by the work of Frohlick and Sack [11], Oldroyd [12] developed many rate type fluid models. Amongst these the one that is amenable to analysis and more importantly experimental corroboration is the Oldroyd-B model. This model is able to describe the stress-relaxation, creep and normal stress differences but it cannot describe shear thinning/shear thickening effects which many polymeric materials exhibit. It has been shown recently by Rajagopal and Srinivasa [13] that an Oldroyd-B fluid is one which stores energy like a linearized elastic solid, its dissipation however being due to two dissipative mechanisms that implies that they arise from a mixture of two viscous fluids [5].

By keeping the above facts in mind our concern in the present investigation is to discuss the decay of a potential vortex in an Oldroyd-B fluid filling the porous space. Such flow analysis appears to be of special interest in oil reservoir engineering, where increasing interest is shown in the possibility of improving oil recovery efficiency from water flooding projects through mobility control with non-Newtonian fluids. Furthermore the flows through porous space occur in packed bed reactors, petroleum engineering, lungs, boiling in porous media and many others. In this communication the modeled governing equation is given in Section 2. Expressions of velocity and tangential tension are presented in Section 3. In Section 4

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1468-1218/$ – see front matter © 2009 Published by Elsevier Ltd
doi:10.1016/j.nonrwa.2008.03.021

some existing solutions are recovered. Section 5 includes the graphical results. The graphs for key parameters are sketched and discussed.

2. Governing equation

The constitutive equations of an Oldroyd-B fluid are

$$\mathbf{T} = -p\mathbf{I} + \mathbf{S}, \tag{1}$$

$$\mathbf{S} + \lambda\left(\frac{\mathrm{d}\mathbf{S}}{\mathrm{d}t} - \mathbf{L}\mathbf{S} - \mathbf{S}\mathbf{L}^T\right) = \mu\left[\mathbf{A}_1 + \lambda_r\left(\frac{\mathrm{d}\mathbf{A}_1}{\mathrm{d}t} - \mathbf{L}\mathbf{A}_1 - \mathbf{A}_1\mathbf{L}^T\right)\right] \tag{2}$$

in which $\mathbf{T}$ is the Cauchy stress, p the hydrostatic pressure, $\mathbf{I}$ the identity tensor, $\mathbf{S}$ the extra stress tensor, $\mathrm{d}/\mathrm{d}t$ the material derivative, $\mathbf{L}$ the velocity gradient, $\mathbf{L}^T$ the transpose of $\mathbf{L}$, μ the dynamic viscosity, t the time, λ the relaxation time and λ_r the retardation time. The expressions for first Rivlin–Ericksen tensor $\mathbf{A}_1$ and $\mathbf{L}$ are

$$\mathbf{A}_1 = \mathbf{L} + \mathbf{L}^T, \quad \mathbf{L} = \nabla V, \tag{3}$$

where ∇ is the gradient operator, $\mathbf{V}$ the velocity and $\lambda > \lambda_r > 0$.

The incompressibility condition and equation of motion with no pressure gradient are respectively given as

$$\operatorname{div}\mathbf{V} = 0, \tag{4}$$

$$\rho\frac{\mathrm{d}\mathbf{V}}{\mathrm{d}t} = \operatorname{div}\mathbf{S} + \mathbf{R}. \tag{5}$$

Here ρ is the fluid density and $\mathbf{R}$ the Darcy resistance in the porous medium.

Our interest lies in the circular motion of an Oldroyd-B fluid through a porous medium therefore the velocity field is

$$\mathbf{V}(r,t) = w(r,t)\,\mathbf{e}_\theta, \tag{6}$$

where $\mathbf{e}_\theta$ is the unit vector along the θ direction and the above equation satisfies Eq. (4) automatically. The initial distribution of the velocity is assumed to be that of a potential vortex of circulation Γ_0 [11] i.e.

$$w(r,0) = \Gamma_0/(2\pi r). \tag{7}$$

Substituting Eq. (6) into Eq. (2) and having in mind the initial condition (the fluid is at rest at $t = 0$)

$$\mathbf{S}(r,0) = \mathbf{0} \tag{8}$$

one has $S_{rr} = S_{rz} = S_{\theta z} = S_{zz} = 0$,

$$\left(1 + \lambda\frac{\partial}{\partial t}\right)S_{\theta\theta} - 2\lambda\left(\frac{\partial w}{\partial r} - \frac{w}{r}\right)S_{r\theta} = -2\mu\lambda_r\left(\frac{\partial w}{\partial r} - \frac{w}{r}\right)^2, \tag{9}$$

$$\left(1 + \lambda\frac{\partial}{\partial t}\right)S_{r\theta} = \mu\left(1 + \lambda_r\frac{\partial}{\partial t}\right)\left(\frac{\partial w}{\partial r} - \frac{w}{r}\right), \tag{10}$$

and the equation of motion gives

$$\frac{1}{r}S_{\theta\theta} = \rho\frac{w^2}{r}, \tag{11}$$

$$\frac{\partial S_{r\theta}}{\partial r} + 2\frac{S_{r\theta}}{r} = \rho\frac{\partial w}{\partial t} + R_\theta. \tag{12}$$

In the porous medium, the constitutive relationship between the pressure drop and velocity for an Oldroyd-B fluid is [4]

$$\left(1 + \lambda\frac{\partial}{\partial t}\right)\nabla p = -\frac{\mu\phi}{k}\left(1 + \lambda_r\frac{\partial}{\partial t}\right)\mathbf{V}, \tag{13}$$

where ϕ is the porosity of the porous medium and k the permeability. Note that for $\lambda = \lambda_r = 0$, Eq. (13) reduces to the classical Darcy's law. Since the pressure gradient in Eq. (13) is a measure of the flow resistance in the bulk of the porous medium and $\mathbf{R}$ in Eq. (5) is interpreted as the flow resistance offered by the solid matrix, therefore, $\mathbf{R}$ satisfies

$$\left(1 + \lambda\frac{\partial}{\partial t}\right)\mathbf{R} = -\frac{\mu\phi}{k}\left(1 + \lambda_r\frac{\partial}{\partial t}\right)\mathbf{V}. \tag{14}$$

From Eqs. (10), (12) and (14), one can write

$$\lambda\frac{\partial^2 w(r,t)}{\partial t^2} + \frac{\partial w(r,t)}{\partial t} = \nu\left(1 + \lambda_r\frac{\partial}{\partial t}\right)\left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\right]w(r,t) - \frac{\nu\phi}{k}\left(1 + \lambda_r\frac{\partial}{\partial t}\right)w(r,t), \tag{15}$$

where ν is the kinematic viscosity.