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# Perturbation techniques for nonexpansive mappings with applications

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ABSTRACT

Perturbation techniques for nonexpansive mappings are studied. An iterative algorithm involving perturbed mappings in a Banach space is proposed and proved to be strongly convergent to a fixed point of the original mapping. These techniques are applied to solve the split feasibility problem and the multiple-sets split feasibility problem, and to find zeros of accretive operators.



## 1. Introduction

Many practical problems can be formulated as a fixed point problem

$$x = Tx, \tag{1.1}$$

where $T$ is a nonexpansive mapping defined on a closed convex subset $C$ of a Hilbert (or Banach, more generally) space $X$. For instance, some problems in signal processing (e.g., phase retrieval [12,26] and design of a nonlinear synthetic discriminant filter for optical pattern recognition (cf. [4]) can be formulated as a split feasibility problem (SFP) [4]:

$$\text{Find a point } x^* \text{ with the property: } x^* \in C \text{ and } Ax^* \in Q, \tag{1.2}$$

where $C$ and $Q$ are closed convex subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, and $A : \mathbb{R}^n \to \mathbb{R}^m$ is a linear operator. The SFP (1.2) can equivalently be rewritten as a fixed point problem [3]

$$x^* = Tx^* := P_C(I - \gamma A^*(I - P_Q)A)x^*, \tag{1.3}$$

where $P_C$ and $P_Q$ are the (nearest point) projections onto $C$ and $Q$, respectively, $\gamma$ is any positive parameter, and $A^*$ is the adjoint of $A$. It is known that for sufficiently small $\gamma > 0$ the mapping $T$ in (1.3) is nonexpansive.

Another example is the intensity-modulated radiation therapy (IMRT) which have received a great deal of attention recently; see [21,28,14,5,6] and references therein (we follow the description in [5,6]).

In IMRT, beams of penetrating radiation are directed at the tumor lesion from external sources. A multileaf collimator (MLC) is used to split each beam into many beamlets with individually controllable intensities. Two problems are pertinent.

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The first one is to calculate the radiation dose absorbed in the irradiated tissue based on a given distribution of beamlet intensities. The second one is the inverse problem of the first one: to find a distribution of radiation intensities (radiation intensity map) deliverable by all beamlets which would result in a clinically acceptable dose distribution; that is, the dose to each tissue should be within the desired upper and lower bounds which are prescribed based on medical diagnosis, knowledge and experience.

The dose distribution is described by the concept of equivalent uniform dose (EUD) which is defined for tumors as the biological equivalent dose that, if given uniformly, will lead to the same cell-kill in the tumor volume as the actual non-uniform dose distribution. The use of EUD was proposed for treatment planning in 1984.

EUD-based IMRT optimization was described as early as in 2002 (see [29]). In such optimization, EUD is formulated as constraints instead of optimizing the EUD functions.

Most clinical constraints are naturally described as constraints on the dose delivered to patients. For instance, the lower and upper dose bounds on each tissue can be described as minimum and maximum dose constraints on the dose delivered to each voxel. EUD constraints are described in the dose space. In contrast, constraints on the deliverable radiation intensities are described in the intensity space. The universal example of this is the requirement of nonnegative beamlet intensity. Additional constraints can be described as limits on the complexity of the intensity map or limits on the acceptable delivery time. Each constraint is described by a set of either dose vectors or intensity vectors. A satisfactory treatment plan would be a plan that fulfill all the constraints; that is, a plan in the intersection of all these constraint sets. Let's formulate these constraints.

Divide the entire volume of the patient into $I$ voxels, indexed by $i = 1, 2, \ldots, I$. Suppose the number of beamlets is $J$, indexed by $j = 1, 2, \ldots, J$. Assume that the radiation is delivered independently from each of the $J$ beamlets. The intensities $x_j$ of the beamlets are arranged in a $J$-dimensional space vector $x = (x_j)_{j=1}^J \in \mathbb{R}^J$, the radiation intensity space.

The quantities $d_{ij} \geq 0$, which represent the dose absorbed in voxel $i$ due to radiation of unit intensity from the $j$-th beamlet, can be calculated by forward calculation program. Let $h_i$ denote the total dose absorbed in voxel $i$ and let $h = (h_i)_{i=1}^I$ be the vector of doses absorbed in all voxels. Thus, $h \in \mathbb{R}^I$, the dose space.

We can calculate $h$ in the following way:

$$h = Ax,$$

where

$$A = [d_{ij}]_{\substack{1\leq i\leq I\\ 1\leq j\leq J}}$$

is the dose influence matrix.

Assume also there are $M$ constraints in the dose space $\mathbb{R}^I$ and $N$ constraints in the intensity space $\mathbb{R}^J$.

Let $H_m$ be the set of dose vectors that fulfill the $m$-th dose constraint, and let $X_n$ be the set of beamlet intensity vectors that fulfill the $n$-th intensity constraint.

In the dose space, typical constraints are that, given a critical structure $S_t$, the dose should exceed neither an upper bound $u_t$ nor a lower bound $l_t$:

$$H_{\max,t} = \{h \in \mathbb{R}^I : h_i \leq u_t,\ i \in S_t\}$$

and

$$H_{\min,t} = \{h \in \mathbb{R}^I : l_t \leq h_i,\ i \in S_t\}.$$

To handle the EUD constraints for each volume of interest $S_t$ consisting of $N_t$ voxels, consider the EUD function

$$E_t(h) = \left(\frac{1}{N_t}\sum_{i\in S_t}(h_i)^{\alpha_t}\right)^{1/\alpha_t}.$$

The parameter $\alpha_t$ is a tissue-specific number which is negative for target volumes and positive for the organs at risk (OAR). For $\alpha_t = 1$, the EUD function is the mean dose of the organ for which it is calculated. On the other hand, letting $\alpha_t \to \infty$ makes the EUD function approach the maximal value: $\max\{h_i : i \in S_i\}$. The EUD constraint for an upper bound $e_t$ for a structure $S_t$ can be described by the set

$$H_{EUD,t} = \{h \in \mathbb{R}^I : E_t(h) \leq e_t\}.$$

The lower EUD bounds are defined similarly.

It has been shown that, due to nonnegativity of the dose, $h \geq 0$, the EUD function $E_t$ is convex for all $\alpha_t \geq 1$ and concave for all $\alpha_t \leq 1$. Therefore, the constraint set $H_{EUD,t}$ is always convex in the dose vector space.

In the radiation intensity space, the most prominent constraint is the nonnegativity of the intensities:

$$X_0 = \{x \in \mathbb{R}^J : x_j \geq 0,\ \ 1 \leq j \leq J\}.$$

In summary, we have formulated the constraints in the radiation intensity space $\mathbb{R}^J$ and in the dose space $\mathbb{R}^I$, respectively, and the two spaces are related by the dose influence matrix $A$. So our problem is formulated as follows:

$$\text{Find an } x^* \in \bigcap_{n=0}^{N} X_n \quad \text{such that } Ax^* \in \bigcap_{j=1}^{M} H_j. \tag{1.4}$$

The formulation of (1.4) is referred to as a multiple-sets split feasibility problem (MSSFP).