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A quasilinearization method for a class of second order singular nonlinear differential equations with nonlinear boundary conditions

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Abstract

We consider the differential equation $-(1/w)(pu')' + \mu u = Fu$, where F is a nonlinear operator, with nonlinear boundary conditions. Under appropriate assumptions on p, w, F and the boundary conditions, the existence of solutions is established. If the problem has a lower solution and an upper solution, then we use a quasilinearization method to obtain two monotonic sequences of approximate solutions converging quadratically to a solution of the equation.

MSC: 34B15; 34A34

Keywords: Nonlinear ordinary differential equations; Nonlinear boundary conditions; Upper and lower solutions; Quasilinearization method

1. Introduction

In this paper we consider the nonlinear differential equation

$$(\ell + \mu)y(t) := -\frac{1}{w(t)}(p(t)y'(t))' + \mu y(t) = Fy(t) \quad \text{a.e. on } I = (0, 1), \tag{1}$$

where F is a nonlinear operator, $p, w > 0$ a.e. on I and μ is a real number. We assume that $1/p$ and w are locally integrable in $(0, 1]$, but $1/p$ is not integrable in any neighborhood of 0. In this case 0 is a singular endpoint for ℓ and 1 is a regular endpoint. We will discuss the problem of existence of solutions for (1) for a class of expressions ℓ, nonlinear operators F and under certain nonlinear boundary conditions. We will also discuss applying the quasilinearization method for the approximation of its solutions. This paper is a generalization of [4] to a class of singular problems. It also generalizes the existence principle in [16]. It turns out that the class of singular problems considered here does not allow us to apply our setting in continuous function spaces. That is why we use L^p spaces instead. These spaces are not algebras, so we use nonlinear operators instead of nonlinear operators defined by classical nonlinear functions such as $f(t, u(t))$, $t \in I$, since second derivatives of these functions do not give rise to continuous bilinear operators on L^p spaces.

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Recently, there has been a lot of research activity on the development and application of the quasilinearization (QSL) method. Nieto [11,12] and Cabada et al. [3] developed and generalized the method for problems with Dirichlet boundary conditions. Ahmad et al. [2,1] studied problems with Neumann boundary conditions, Jankowski [5] studied the method for problems with nonlinear boundary conditions, and Lakshmikantham and Vatsala [8] compared the QSL with Newton method.

The QSL method gives excellent results when applied to different nonlinear ordinary differential equations in physics, such as the Blasius, Duffing, Lane–Emden and Thomas–Fermi equations [9]. Also, recently the QSL technique has been applied to study a medical problem (a biomathematical model of blood flow inside an intracranial aneurysm) [6,13–15]. For more on the method and its applications, see the manuscript [7].

This paper contains two sections besides the introduction. In Section 2, we discuss a general existence theorem for the class of problems considered. In Section 3, we discuss the existence and quasilinearization under the assumption that our problem has a lower solution and an upper solution.

2. Existence of solutions

In this section we develop an existence theorem for a class of nonlinear problems of the form (1). We begin by specifying assumptions on ℓ.

(A1) w, $R^{1/\sigma}/p \in L^1(I)$, where $R(s)=\int_0^s w(x)\,\mathrm{d}x$ and $\sigma \geqslant 1$.

We will use $y^{[1]}(s)$ to denote the pseudo derivative [10] $py'(s)$.

Theorem 1. *Assume* (A1) *holds*. *Then*:

1. *The problem*

$$\begin{aligned}&(\ell+\mu)y=0,\\&y(0)=a\neq 0,\quad y^{[1]}(0^+)=0\end{aligned}\tag{2}$$

has a solution $y_1 \in C[0,1]$ *such that* $y_1^{[1]} \in AC[0,1]$.

2. *The problem*

$$\begin{aligned}&(\ell+\mu)y=0,\\&y(1)=0,\quad y^{[1]}(1)=1\end{aligned}\tag{3}$$

has a solution $y_2 \in L_w^\sigma(0,1)\cap C(0,1]$ *with* $y_2^{[1]} \in AC[0,1]$.

Proof. The case $\sigma>1$ was proved in [16], so we prove the case $\sigma=1$ here.

(1): Let $K>0$ and define the norm $\|\cdot\|_K$ on $C[0,1]$ by

$$\|u\|_K=\sup_{t\in[0,1]}\{\mathrm{e}^{-KR(t)}u(t)\}.$$

Solving Eq. (2) is equivalent to finding $y\in C[0,1]$ such that

$$y(t)=a-\mu\int_0^t\frac{1}{p(s)}\int_0^s y(x)w(x)\,\mathrm{d}x.$$

Define the operator $N: C[0,1]\to C[0,1]$ by

$$Ny(t)=a-\mu\int_0^t\frac{1}{p(s)}\int_0^s y(x)w(x)\,\mathrm{d}x\,\mathrm{d}s.$$