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Existence and uniqueness of periodic solutions for a kind of second order neutral functional differential equations☆

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Abstract

In this paper, we use the coincidence degree theory to establish new results on the existence and uniqueness of T-periodic solutions for the second order neutral functional differential equation of the form

$$(x(t) + Bx(t-\delta))'' + Cx'(t) + g(x(t-\tau(t))) = p(t).$$

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1. Introduction

Consider the second order neutral functional differential equation of the form

$$(x(t) + Bx(t-\delta))'' + Cx'(t) + g(x(t-\tau(t))) = p(t), \tag{1.1}$$

where $\tau, p, g : R \to R$ are continuous functions, B, δ and C are constants, τ and p are T-periodic, $C \neq 0$, $|B| \neq 1$ and $T > 0$. In recent years, the problem of the existence of periodic solutions for some types of the second order functional differential equation, especially for delay Duffing equation and delay Liénard equation, has been extensively studied in the literature. We refer the reader to [1,4–7] and the references cited therein. However, to the best of our knowledge, there exist no results for the existence and uniqueness of periodic solutions of Eq. (1.1).

The main purpose of this paper is to establish sufficient conditions for the existence and uniqueness of T-periodic solutions of Eq. (1.1). The results of this paper are new and they complement previously known results. An illustrative example is given in Section 4.

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For ease of exposition, throughout this paper we will adopt the following notations:

$$|x|_k = \left(\int_0^T |x(t)|^k \,\mathrm{d}t\right)^{1/k}, \quad |x|_\infty = \max_{t\in[0,T]} |x(t)|.$$

Let

$$X = \{x | x \in C^1(R, R), x(t+T) = x(t), \text{ for all } t \in R\}$$

and

$$Y = \{x | x \in C(R, R), x(t+T) = x(t), \text{ for all } t \in R\}$$

be two Banach spaces with the norm

$$\|x\|_X = \max\{|x|_\infty, |x'|_\infty\} \quad \text{and} \quad \|x\|_Y = |x|_\infty.$$

Define linear operators A and L in the following form respectively:

$$A : Y \longrightarrow Y, \qquad A(x(t)) = x(t) + Bx(t-\delta)$$

and

$$L : D(L) \subset X \longrightarrow Y, \qquad Lx = (Ax)'', \tag{1.2}$$

where $D(L) = \{x | x \in X, x'' \in C(R, R)\}$.

We also define a nonlinear operator $N : X \longrightarrow Y$ by setting

$$Nx = -Cx'(t) - g(x(t-\tau(t))) + p(t). \tag{1.3}$$

Since $|B| \neq 1$, from Lemma 1 in [5], it follows that

(C_1) A has a unique continuous bounded inverse A^{-1} on Y;

(C_2) Let $x(t) + Bx(t-\delta) = A(x(t)) = f(t)$, we have

$$x(t) = A^{-1}(f(t)) = \begin{cases} \sum_{j\geqslant 0} (-B)^j f(t-j\delta), & |B| < 1, \\ -\sum_{j\geqslant 0} (-B)^{-j-1} f(t+j\delta), & |B| > 1. \end{cases}$$

Meanwhile, according to (C_1) and (C_2), it is easy to see that $(Ax)'' = Ax''$, for all $x \in D(L)$. Therefore,

$$\mathit{Ker}\, L = R \quad \text{and} \quad \operatorname{Im} L = \left\{ x \,\middle|\, x \in C_T, \int_0^T x(s)\,\mathrm{d}s = 0 \right\}.$$

Thus, the operator L is a Fredholm operator with index zero.

Define the continuous projectors $P : X \longrightarrow \mathit{Ker}\, L$ and $Q : Y \longrightarrow Y/\operatorname{Im} L$ by setting

$$Px(t) = x(0) = x(T)$$

and

$$Qx(t) = \frac{1}{T}\int_0^T x(s)\,\mathrm{d}s.$$

Hence, $\operatorname{Im} P = \mathit{Ker}\, L$ and $\mathit{Ker}\, Q = \operatorname{Im} L$. Set $L_P = L|_{D(L)\cap \mathit{Ker}\, P}$, then L_P has continuous inverse L_P^{-1} defined by

$$\begin{aligned} &L_P^{-1} : \operatorname{Im} L \longrightarrow D(L) \cap \mathit{Ker}\, P,\ L_P^{-1} y(t) \\ &\quad = A^{-1}\left(-\frac{t}{T}\int_0^T (t-s)y(s)\,\mathrm{d}s + \int_0^t (t-s)y(s)\,\mathrm{d}s\right). \end{aligned} \tag{1.4}$$