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Algebro-geometric solutions for some (2 + 1)-dimensional discrete systems

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Abstract

Starting from a discrete spectral problem, a discrete soliton hierarchy is derived. Some (2 + 1)-dimensional discrete systems related to the hierarchy are proposed. The elliptic coordinates are introduced and the equations in the discrete soliton hierarchy are decomposed into solvable ordinary differential equations. The straightening out of the continuous flow and the discrete flow are exactly given through the Abel–Jacobi coordinates. As an application, explicit algebro-geometric solutions for the (2 + 1)-dimensional discrete systems are obtained.

Keywords: (2 + 1)-dimensional discrete systems; Algebro-geometric solutions; Elliptic coordinates; Straightening out; Riemann–Jacobi inversion technique

1. Introduction

There have been several systematic approaches to obtain explicit solutions of the soliton equations, such as the inverse scattering transformation, the Bäcklund transformation, the algebro-geometric method, the polar expansion method and so on [2,3,11–14]. Some interesting explicit solutions have been found, for example, pure-soliton solutions, quasi-periodic solutions, polar expansion solutions, etc. The algebro-geometric method was first developed by Matveev, Its, Novikov et al. as analog of inverse scattering theory [6,10,5]. This method allows us to find an important class of exact solutions to the soliton equations. As a degenerated case of this solutions, the multisoliton solutions and elliptic functions may be obtained [12]. Recently, based on the nonlinearization technique of Lax pairs, algebro-geometric solutions for (1 + 1)-dimensional and (2 + 1)-dimensional soliton equations have been obtained by Cao and Geng [4,7,8].

In recent years, the study of nonlinear integrable lattice equations has become the focus of common concern in the theory of integrable systems. Many nonlinear integrable lattice equations have been proposed and discussed, for example, the Ablowitz–Ladik lattice [1], the Toda lattice [18], and so on. In this paper, we will consider a discrete spectral problem

$$E\psi(n) = U_n\psi(n) = \begin{pmatrix} \lambda^{-1}(1+p_nq_n) & p_n \\ q_n & \lambda \end{pmatrix}\psi(n), \qquad (1.1)$$

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where E is the shift operator, $Ef(n)=f(n+1)$. In Section 2, we will derive a hierarchy of lattice soliton equations from (1.1). We will also propose Some $(2+1)$-dimensional differential-difference equations related to the discrete soliton hierarchy. In Section 3, based on finite-order expansion of the Lax matrix, we introduce elliptic coordinates. The spectral solutions of the differential-difference equations are reduced to solving ordinary differential equation. In Sections 4 and 5, the Abel–Jacobi coordinates are introduced, by which the straightening out of the continuous flow and the discrete flow are studied in detail. In Section 6, the Riemann–Jacobi inversion is discussed, from which the algebro-geometric solutions for the $(2+1)$-dimensional differential-difference equations are obtained by using the Riemann theta functions.

2. The discrete soliton hierarchy

In order to derive the hierarchy related to (1.1), we first introduce Lenard's gradient sequence S_j, $0\leqslant j\in Z$, by the recursion equation

$$K_nS_j(n)=J_nS_{j+1}(n),\quad J_nS_0(n)=0,\quad j\geqslant 0 \tag{2.1}$$

with two operators

$$K_n=\begin{pmatrix}(1+p_nq_n)E & 0 & 0\\ 0 & 1+p_nq_n & 0\\ -p_n & q_nE & (1+p_nq_n)(E-1)\end{pmatrix},$$

$$J_n=\begin{pmatrix}1 & 0 & q_n(E+1)\\ 0 & E & p_n(E+1)\\ -p_n & q_nE & (1+p_nq_n)(E-1)\end{pmatrix}.$$

Equation $J_nS_0(n)=0$ has a special solution

$$S_0(n)=\left(q_n,\ p_{n-1},\ -\tfrac{1}{2}\right)^{\mathrm{T}} \tag{2.2}$$

and we have

$$ker\, J_n=\{cS_0(n)\mid \forall c\ (constant)\}.$$

Then $S_j(n)$ is uniquely determined by the recursion relation (2.1) up to a term $cS_0(n)$, which is always assumed to be zero. The first few numbers are

$$S_1(n)=\begin{pmatrix}q_{n+1}-p_{n-1}q_n^2\\ p_{n-2}-p_{n-1}^2q_n\\ p_{n-1}q_n\end{pmatrix},$$

$$S_2(n)=\begin{pmatrix}q_{n+2}-p_nq_{n+1}^2-2p_{n-1}q_nq_{n+1}-p_{n-2}q_n^2+p_{n-1}^2q_n^3\\ p_{n-3}-p_{n-2}^2q_{n-1}-2p_{n-2}p_{n-1}q_n-p_{n-1}^2q_{n+1}+p_{n-1}^3q_n^2\\ p_{n-1}q_{n+1}+p_{n-2}q_n-p_{n-1}^2q_n^2\end{pmatrix}.$$

Assume that the time dependence of $\psi(n)$ for the spectral problem (1.1) is

$$\psi(n)_{t_m}=V_n^{(m)}\psi(n),\quad V_n^{(m)}=\begin{pmatrix}A_n^{(m)} & B_n^{(m)}\\ C_n^{(m)} & -A_n^{(m)}\end{pmatrix} \tag{2.3}$$