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Study on the stability of nonlinear differential equations with initial time difference[☆]

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ABSTRACT

This paper establishes several criteria of stability for nonlinear differential dynamical systems relative to initial time difference by employing perturbing Lyapunov functions. The reported novel results are more easy to be tested than the existing results. As an application, these results are applied to a nonlinear dynamical system to get the stability properties relative to initial time difference.

1. Introduction

The stability theory in the sense of Lyapunov has gained extensive applications, but the various definitions of stability in the sense of Lyapunov assume that the initial value is unchanged, whereas in the real world, it is impossible not to make errors in the starting time because of all kinds of disturbed factors. So it is reasonable to study the solutions of the differential equation with variation in the initial time. How to compare any two solutions which differ in the initial starting time has a great realistic significance.

There are two ways of comparing and measuring the difference of two solutions. For instance, in [1–4], the method of variation of parameters is used to discuss such situations in one direction. In [5–12], the authors have obtained several stability results for differential and integrodifferential equations by using a differential inequalities technique. In [2], Lakshmikantham has proposed the definition of practical stability for nonlinear differential equations with initial time difference and given a practical stability criterion. In [1,9], the authors proposed the definitions of uniform stability and uniformly asymptotical stability and proved two stability criteria, but they did not discuss asymptotical stability. McRae in [11] has given the definitions of equistability and equi-asymptotical stability and presented two stability criteria, but these are not satisfactory because the authors only obtain stability criteria under rigid conditions, and it is difficult to find perturbing Lyapunov functions satisfying all the constraints. The more complex the considered dynamical systems, the more difficult it is to find a Lyapunov function to satisfy all the desired conditions. In those situations, one may find it more advantageous to perturb that Lyapunov function as opposed to discarding it. The method of perturbing Lyapunov functions offers a very flexible mechanism since each function can satisfy less rigid requirements [13].

In this paper, we present a new comparison principle for the solution of nonlinear differential equations with the initial time changed. Using the comparison principle we present several new stability criteria of nonlinear differential

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equations with initial time difference. We obtain sufficient conditions for various types of stability of nonlinear differential systems relative to initial time difference by using perturbing Lyapunov functions. We also construct a nonlinear differential dynamical system and apply our results to get the equistability properties of the solutions of the dynamical system relative to initial time difference.

The remainder of this paper is organized as follows. In Section 2, we introduce the notations used throughout the paper and the definitions of stability, and we also present two useful lemmas in our investigations. In Section 3, we present some results for various types of stability of nonlinear differential system relative to initial time difference. In Section 4, we give an example as an application of these results. Finally, the conclusions are given in Section 5.

2. Preliminaries

Consider the differential system in a real n-dimensional Euclidean space with norm $\|\cdot\|$,

$$x' = f(t, x) \tag{1}$$

where $f \in C[R_+ \times R^n, R^n]$. Let $x^*(t) = x^*(t, t_0, y_0)$ and $x(t) = x(t, \tau_0, x_0)$ be the solutions of (1) through (t_0, y_0) and (τ_0, x_0), respectively. Suppose that $x^*(t)$ is the given solution relative to which we shall study the stability criteria. Let $\eta = \tau_0 - t_0 > 0$ and denote $S(x^*, \rho) = \{x \in R^n : \|x - x^*(t)\| < \rho\}$, $S(\rho) = \{x \in R^n : \|x\| < \rho\}$ for some $\rho > 0$. Let $S^c(x^*, \rho)$ and $S^c(\rho)$ be the complementary sets of $S(x^*, \rho)$ and $S(\rho)$, respectively. We define the following classes of functions: $K = \{a \in C(R_+, R_+) : a \text{ is strictly increasing and } a(0) = 0\}$.

$C_K = \{a \in C(R_+^2, R_+) : a(t, u) \in K \text{ for each } t \in R_+\}$.

We shall introduce the following definitions of stability relative to initial time difference to nonlinear dynamical systems.

Definition 2.1 (*See [11]*). The solution $x^*(t) = x^*(t, t_0, y_0)$ of system (1) is said to be

(S_1) equistable, if given $\epsilon > 0$ and $t_0 \in R_+$, there exist $\delta = \delta(t_0, \epsilon) > 0$ and $\sigma = \sigma(t_0, \epsilon) > 0$ such that $\|x_0 - y_0\| < \delta$, $|\eta| < \sigma$ implies $\|x(t + \eta, \tau_0, x_0) - x^*(t, t_0, y_0)\| < \epsilon$, $t \geq t_0$;

(S_2) uniformly stable if (S_1) holds with δ and σ independent of $t_0 \in R_+$;

(S_3) equi-asymptotically stable, if (S_1) holds and given $\epsilon > 0$ and $t_0 \in R_+$, there exist $\delta_0 = \delta_0(t_0) > 0$, $\sigma_0 = \sigma_0(t_0) > 0$ and a $T = T(t_0, \epsilon) > 0$ such that $\|x_0 - y_0\| < \delta_0$, $|\eta| < \sigma_0$ implies $\|x(t + \eta, \tau_0, x_0) - x^*(t, t_0, y_0)\| < \epsilon$, $t \geq t_0 + T$;

(S_4) uniformly asymptotically stable if (S_2) and (S_3) holds with δ_0, σ_0 and T in (S_3) independent of t_0.

Considering the nonlinear dynamical system (1), for $V \in C[R_+ \times R^n, R_+^m]$, we define the generalized derivative with respect to system (1):

$$D_-V(t, x - y) = \liminf_{h \to 0^-} \frac{1}{h}[V(t + h, x - y + h(f(t, x) - f(t, y))) - V(t, x - y)]. \tag{2}$$

We note that if $V \in C^1[R_+ \times R^n, R_+^m]$, then $D_-V(t, x - y) = V'(t, x - y) = V_1'(t, x - y) + V_2'(t, x - y)^{\mathrm{T}}(f(t, x) - f(t, y))$.

In this section, we shall present two comparison principles, which will be very useful in our investigations.

Lemma 2.1 (*See [14]*). *Assume that*

(a) $m \in C[R_+, R_+]$, $g \in C[R_+^2, R]$ *and* $D_-m(t) \leq g(t, m(t))$, $t \in R_+$, *where*

$$D_-m(t) = \liminf_{h \to 0^-} \frac{1}{h}[m(t + h) - m(t)];$$

(b) $r(t) = r(t, t_0, u_0)$ *is the maximal solution of* $u' = g(t, u)$, $u(t_0) = u_0 \geq 0$, $t_0 \geq 0$ *existing on* $[t_0, \infty)$.

Then $m(t_0) \leq u_0$ *implies* $m(t) \leq r(t)$, $t \geq t_0$.

Lemma 2.2 (*See [2]*). *Assume that*

(a) $m \in C[R_+, R_+]$, $g \in C[R_+^2, R]$ *and* $D_-m(t) \leq g(t, m(t))$, $m(t_0) \leq w_0$, $t_0 \geq 0$, *where*

$$D_-m(t) = \liminf_{h \to 0^-} \frac{1}{h}[m(t + h) - m(t)];$$

(b) *the maximal solution* $r(t) = r(t, \tau_0, w_0)$ *of* $w' = g(t, w)$, $w(\tau_0) = w_0 \geq 0$, $\tau_0 \geq 0$, *exists for* $t \geq \tau_0$;

(c) $g(t, w)$ *is nondecreasing in* t *for each* w *and* $\tau_0 > t_0$.

Then $m(t) \leq r(t + \eta)$, $t \geq t_0$ *and* $m(t - \eta) \leq r(t)$, $t \geq \tau_0$.