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The approximate solution of steady temperature distribution in a rod: Two-point boundary value problem with higher order nonlinearity

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ABSTRACT

In this paper, two-point boundary value problems have been solved by the well-known variational iteration method. Considering the situation in which the nonlinear part is a polynomial function with degree of ≥ 2, the steady temperature distribution in a rod has been computed. The strongly nonlinear differential equation has been become a reduced differential equation by the aid of a proper transformation and variational iteration method has been applied to the boundary value problem.

1. Introduction

The steady state temperature distribution in a rod of length L is considered by Yuan-Ming Wang [1] as the two point boundary value problem

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(k(T)\frac{\mathrm{d}T}{\mathrm{d}x}\right) = f(x, T), \quad 0 \leq x \leq L, \tag{1}$$

subject to boundary conditions $T(0) = \alpha$ and $T(L) = \beta$

where one end of rod is held at a given constant temperature α and the other is maintained at a given constant temperature β, function $f(x, T)$ is sufficiently smooth, and function $k(T)$ depends on the temperature distribution, and represents the thermal conductivity. Thermal conductivities of materials change with temperature while for most gases at moderate pressures it is a function of temperature alone, to be more complicated, therefore, the functions $k(T)$, $f(x, T)$ are chosen nonlinear functions with respect to T. For the above problem, it is not easy to calculate its analytic solution but the numerical methods are able to be used mostly for solving those complex heat transfer problems involving many repetitive calculations with changing boundary conditions. Wang used a transformation that transforms (1) into a feasible second order boundary value problem in order to apply Numerov's method developed by Agarwal and Wang in [2].

In this paper, assuming that we are also interested in the general form of the two point boundary value problem (1), we investigate the way to find the steady temperature distribution in a rod of length L. For this purpose we pay attention the variational iteration method to calculate its exact solution. The advantage of the chosen method is that its celerity reaches the approximate solution.

Especially if one is taken $k(T) \equiv 1$, (1) becomes

$$\frac{\mathrm{d}^2T}{\mathrm{d}x^2} = f(x, T), \quad 0 \leq x \leq L, \qquad T(0) = \alpha, \quad T(L) = \beta \tag{2}$$

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which can be solved by the Numerov method [1,2], Adomian decomposition method, differential transform method [3] etc., but here it is solved by the variational iteration method [4–11,3,12–14] for which $k(T)$ can be chosen as a linear or nonlinear function of T. The problem (1) becomes different types that are required to make it in a suitable differential equation form in order to solve it by the variational iteration method.

2. Inversion principle

In order to derive the suitable form of differential equation (1) for applying the variational iteration method, (1) will be transformed to a proper problem by making use of the inversion principle. Let F be a continuous function on $\mathbb{R}$ with inverse F^{-1} defined by

$$F^{-1}(v) = \{T : F(T) = v, v_0 < v < v_1\}. \tag{3}$$

The interval I is a subset of $\mathbb{R}$ including the end points α and β of the problem. The function $k(T)$ is a continuous function of T in I and there exist positive constants k_0 and k_1 such that $k_0 \leq k(T) \leq k_1$ in I [1]. Then the transformation F is

$$v = F(T) = \int_{\rho}^{T} k(s)\mathrm{d}s \quad \text{and} \quad \forall u \in I. \tag{4}$$

Since F is a strictly monotone increasing function of T in I, F^{-1} exists. Using inverse transformation $T = F^{-1}(v)$, problem (1) is transformed in to nonlinear second order boundary value problem:

$$\frac{\mathrm{d}^2 v}{\mathrm{d}x^2} = f(x, F^{-1}(v)), \quad 0 \leq x \leq L, \tag{5}$$
subject to boundary conditions $v(0) = F(\alpha)$ and $v(L) = F(\beta)$.

If the above problem is solved for $v = F(T)$, then by inverse transformation Problem 1 would be solved for T.

3. Application process of variational iteration method to the transformed problem

The function $f(x, F^{-1}(v))$ can be taken as the function $g(x, v)$. The effective variational iteration method is used to solve a wide class of differential equations [3–9] and its methodology is also given from ordinary differential equations to partial differential equations in [10–14]. Here we apply this method to two point boundary value problems in order to compute their solutions that are very closed to analytic solutions. The basic process of the method is to take

$$G\left(x, v, v''\right) = 0 \tag{6}$$

where x is the independent variable, v is the unknown function and G has both linear part Lv and nonlinear part Nv. The correction functional of (6) is

$$v_{n+1}(x) = v_n(x) + \int_0^x \lambda(\xi)\, G\left(\xi, v_n, v''_n\right) \mathrm{d}\xi \tag{7}$$

and

$$v_{n+1}(x) = v_n(x) + \int_0^x \lambda(\xi) \left(v''_n(\xi) - g(\xi, v_n)\right) \mathrm{d}\xi. \tag{8}$$

If $g(x, v)$ is separable of variables such that ψ is a function of x and ϕ is a function of v, (8) can be written as either

$$v_{n+1}(x) = v_n(x) + \int_0^x \lambda(\xi) \left(v''_n(\xi) - \psi(\xi)\phi(v_n)\right) \mathrm{d}\xi \tag{9}$$

or

$$v_{n+1}(x) = v_n(x) + \int_0^x \lambda(\xi) \left(v''_n(\xi) - \psi(\xi) - \phi(v_n)\right) \mathrm{d}\xi. \tag{10}$$

Especially, if the function $\phi(v)$ is a continuous differentiable nonlinear function in the interval that we are interested, that is, a polynomial function degree of ≥ 2 with respect to v; then making stationary (9) and (10) and noticing that $\delta\tilde{v}_n$ is restricted variation, (9) and (10) can be written as follows:

$$\delta v_{n+1}(x) = \delta v_n(x) + \delta \int_0^x \lambda(\xi) \left(v''_n(\xi) - \psi(\xi)\phi(v_n)\right) \mathrm{d}\xi, \tag{11}$$

and

$$\delta v_{n+1}(x) = \delta v_n(x) + \delta \int_0^x \lambda(\xi) \left(v''_n(\xi) - \psi(\xi) - \phi(v_n)\right) \mathrm{d}\xi, \tag{12}$$

respectively. The following corollary is written as a result for the general situation which will be any polynomial function degree of greater than one.