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Analysis of the nonlinear vibration of a two-mass–spring system with linear and nonlinear stiffness

SHA. Hashemi Kachapi [a,b,*], Rao V. Dukkipati [c], S.Gh. Hashemi K. [d], S.Mey. Hashemi K. [e], S.Meh. Hashemi K. [e], SK. Hashemi K. [f]

[a] *Islamic Azad University, Sari Branch, Department of Mechanical Engineering, Sari, Iran*
[b] *Babol Noshirvani University of Technology, Department of Mechanical Engineering, Babol, Iran*
[c] *Fairfield University, Department of Mechanical Engineering, Fairfield, USA*
[d] *University of Payame Noor Kordkoy Branch, Department of Mathematical, Kordkoy, Iran*
[e] *Hadaf Institute of Technology, Department of Electronic Engineering, Sari, Iran*
[f] *Shomal University, Department of Electronic Engineering, Amol, Iran*

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ABSTRACT

An analytical approach is developed for areas of nonlinear science such as the nonlinear free vibration of a conservative, two-degree-of-freedom mass–spring system having linear and nonlinear stiffnesses. The main contribution of this research is twofold. First, it introduces the transformation of two nonlinear differential equations for a two-mass system using suitable intermediate variables into a single nonlinear differential equation and, more significantly, the treatment of a nonlinear differential system by linearization coupled with Newton's method. Secondly, the major section is the solving of the governing nonlinear differential equation where the displacement of the two-mass system can be obtained directly from the linear second-order differential equation using a first-order variational approach. The aforementioned approach proposed by J.H. He, who actually developed the method, is exactly He's variational method. This approach is an explicit method with high validity for resolving strong nonlinear oscillation system problems. Two examples of nonlinear two-degree-of-freedom mass–spring systems are analyzed, and verified with published results and exact solutions. The method can be easily extended to other nonlinear oscillations and so could be widely applicable in engineering and science.

1. Introduction

The motion of the nonlinear two-degree-of-freedom (TDOF) oscillation system has been widely investigated in the past few decades [1–8]. TDOF systems are important in engineering because many practical engineering components consist of coupled vibrating systems that can be modeled using two-degree-of-freedom systems such as elastic beams supported by two springs and vibration of a milling machine [9]. The equations of motion for nonlinear TDOF systems consist of two second-order differential equations with cubic nonlinearities. Recently, solving the equations of motion for a mechanical system associated with linear and nonlinear properties was attempted through transformation into a set of differential algebraic equations using intermediate variables; these may be further converted into a nonlinear ordinary differential

* Corresponding author at: Islamic Azad University, Sari Branch, Department of Mechanical Engineering, Sari, Iran.
E-mail addresses: h_hashemi_kachapi@yahoo.com, hashemikachapi@iausari.ac.ir (SHA. Hashemi Kachapi), Rdukkipati@mail.fairfield.edu, Rdukkipati@stagweb.fairfield.edu (R.V. Dukkipati).

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equation [10]. Such intermediate variables are introduced here to transform the equations of motion for a TDOF system into the Duffing equation.

One of the intermediate variables is obtained by solving the Duffing equation and the other one is computed from a simple linear second-order differential equation.

Due to the limitation of existing exact solutions, many analytical and numerical approaches have been developed. The harmonic balance method, Galerkin's method and the elliptic Krylov–Bogoliubov method with Jacobi elliptic functions are useful techniques for quantitative analysis of TDOF oscillation systems. The Duffing equation is one of the familiar nonlinear equations which can be solved using different kinds of analytical approximations, including the perturbation method [11], parameter expansion method [12] and other methods [13–15], which are, in some cases, applicable to nonlinear oscillation systems even for rather large amplitude of oscillation. However, it is usually difficult to achieve higher order analytic approximations by using these methods. Although the weighted linearization method [16], the modified Lindstedt–Poincaré method [17], the power series approach [18], the homotopy analysis method [19] and the Taylor series method [20] have been applied to obtain approximate periods of nonlinear equation with large amplitude of oscillation recently, these methods involve sophisticated derivations and computations, and they are difficult to implement.

Variational methods have been, and continue to be, popular tools for nonlinear analysis. When contrasted with other approximate analytical methods, variational methods combine the following two advantages: (1) they provide physical insight into the nature of the solution of the problem; (2) the solutions obtained are the best among all the possible trial functions. Recently, some approximate variational methods, including the approximate energy method [21,22], the variational iteration method [23,24] and the new coupled variational approach which was proposed by SHA. Hashemi Kachapi [25], and the soliton solution, bifurcation, limit cycle and period solutions of nonlinear equations have been given much attention. In the present paper, we use a variational approach for nonlinear oscillators which was proposed by J.H. He [26]. This approach can be seen as a Ritz-like method and leads to a very rapid convergence of the solution, and can be easily extended to other nonlinear oscillations. In short, this approach yields extended scope of applicability, simplicity, flexibility in application, and avoidance of complicated numerical and analytical integration as compared to others among the previous approaches such as the perturbation methods, and so could prove widely applicable in engineering and science; recently, extensive studies have been done on engineering problems by using the variational approach [27–32].

In this paper, two practical examples [6,7] are selected for evaluating and verifying the excellence and accuracy of the proposed method. The accuracy of the proposed approach for solving TDOF systems is compared with published references and exact solutions. The major advantage of the analytical approximation over the numerical integration scheme is the availability of an all-encompassing understanding of the nature of systems in response to changes of parameters affecting nonlinearity. For instance, the perturbation method and variational method are common but these techniques have restrictions that confine their scope of application. The proposed method presented here unrestrictedly provides analytical expressions for the motions of two masses connected by linear and nonlinear spring stiffnesses.

2. Procedures of solution for the Duffing equation using a variational approach

The conservative autonomous system of a cubic Duffing equation is represented by the following second-order differential equation:

$$\ddot{v}(t) + \alpha v(t) + \beta v(t)^3 = 0 \tag{1}$$

with initial conditions

$$v(0) = A, \qquad \dot{v}(0) = 0, \tag{2}$$

in which v and t are generalized dimensionless displacement and time variables, respectively, and α and β are any positive constant parameters.

In the present paper, we consider a general nonlinear oscillator in the form [26]

$$\ddot{v}(t) + f(v(t)) = 0. \tag{3}$$

Its variational principle can be easily established using the semi-inverse method [26]:

$$J(v) = \int_0^{T/4} \left(-\frac{1}{2}\dot{v}^2 + F(v)\right) \mathrm{d}t \tag{4}$$

where $T = 2\pi/\omega$ is the period of the nonlinear oscillator. Using Eq. (4) and $F(v) = \int(\alpha v + \beta v^3)\mathrm{d}v$ yields

$$J(v) = \int_0^{T/4} \left(-\frac{1}{2}\dot{v}^2 + \frac{1}{2}\alpha v^2 + \frac{1}{4}\beta v^4\right) \mathrm{d}t. \tag{5}$$

Nonlinear oscillations systems are not dependent upon initial conditions. In general, oscillation systems contain two important physical parameters, i.e. the frequency ω and the amplitude A of oscillation. So, without any loss of generality, consider some initial conditions:

$$v(0) = A, \quad \dot{v}(0) = 0. \tag{6}$$