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Existence of energy maximizing vortices in a three-dimensional quasigeostrophic shear flow with bounded height

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ABSTRACT

The existence of an energy maximizer relative to a class of rearrangements of a given function is proved. The maximizers are stationary and stable solutions of the quasigeostrophic equation, which governs the time evolution of large-scale three-dimensional geophysical flow in a vertically bounded domain. The background flow is unidirectional, with linear horizontal shear. The theorem proved implies the existence of a family of stationary and stable vortices that rotate in the same direction as the background shear. It extends an earlier theorem by Burton and Nycander, which is valid for a vertically unbounded domain.

1. Introduction

In this paper we will prove the existence of maximizers for a variational problem associated with large-scale geophysical flows. The basic equation governing such flows is the three-dimensional barotropic vorticity equation

$$\frac{\partial Q}{\partial t} + J(Q, \Psi) = 0. \tag{1}$$

Here $\Psi(t, x, y, z)$ is the stream function, J denotes the Jacobian, i.e.

$$J(f, g) = \frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial g}{\partial x}\frac{\partial f}{\partial y}$$

and Q is defined by

$$Q = -\Delta\Psi,$$

where Δ is the three-dimensional Laplacian. Usually Q is called the *potential vorticity* (PV), with positive Q corresponding to cyclonic PV. The flow domain is infinite in the *xy*-plane, but bounded in the vertical, with the boundary conditions

$$\frac{\partial \Psi}{\partial z} = 0, \quad \text{at } z = 0 \text{ and } z = H. \tag{2}$$

According to Eq. (1), Q is advected by a velocity field which is horizontally divergence-free. Therefore, $Q(\cdot, z, t)$ remains in the set of rearrangements of the initial condition $Q(\cdot, z, 0)$. In the terminology of Burton and Nycander [1], $Q(x, y, z, t)$ is a *stratified rearrangement* of $Q(x, y, z, 0)$. The dynamics defined by Eq. (1) also conserve the energy (to be defined below).

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A field Q that maximizes or minimizes the energy in a set of stratified rearrangements therefore defines a stationary and stable flow.

This variational principle was used by Burton and Nycander [1] to prove the existence of a family of stationary and stable vortex solutions in a unidirectional background flow with linear horizontal and vertical shear. However, they assumed that the domain was vertically unbounded, while in virtually all applications the domain is bounded both from above and from below. Their result is therefore relevant only for vortices whose height is very small compared to the height of the atmosphere or the depth of the ocean. Since their proof relied on the convexity of the energy functional, it could not easily be extended to the bounded case, in which it has not been possible to decide whether the energy functional is convex.

In the present paper we extend the result of Burton and Nycander [1] to the more realistic case of a vertically bounded domain. By extending Theorem 3 in Burton [2] to stratified rearrangements, we are able to do this without using convexity. Also, since the boundary condition at the vertical boundaries is not satisfied by a flow with non-vanishing vertical shear, we assume that the shear of the background flow is purely horizontal.

Corresponding results for two-dimensional flow, proving the existence of stable vortices in a background shear flow, have earlier been obtained by Nycander [3] and Emamizadeh [4]. Similar existence proofs, based on a variational principle and rearrangement theory, have also been obtained for three-dimensional vortex rings [2], two-dimensional vortex couples [5], and vortices attached to a localized seamount [6,7]. A heuristic discussion of several of these cases has been given by Nycander [8].

2. Notation and definitions

Assuming that the background flow is unidirectional and has linear horizontal shear, we set

$$\Psi = \psi - c_0 y^2.$$

Substituting this expression into Eq. (1), we obtain

$$\frac{\partial q}{\partial t} + J(q, \psi - c_0 y^2) = 0, \tag{3}$$

where q is the PV anomaly associated with the vortex, and

$$q = -\Delta\psi. \tag{4}$$

Eq. (3) conserves the energy, defined by

$$E(q) = \frac{1}{2}\int_\Omega qKq\,\mathrm{d}r - \int_\Omega c_0 y^2 q(r)\,\mathrm{d}r, \tag{5}$$

where $r = (x, y, z)$, $\Omega = \mathbb{R}^2 \times (0, H)$ and the operator K is defined by

$$Kq(r) = \frac{1}{4\pi}\int_\Omega G(r, r')q(r')\,\mathrm{d}r' \quad \text{for } r \in \Omega.$$

Here $G(r, r')$ denotes the Green's function for $-4\pi\Delta$ with homogeneous Neumann boundary condition on Ω, defined as follows:

$$\begin{aligned} G(r, r') = {} & \frac{1}{[(x-x')^2 + (y-y')^2 + (z-z')^2]^{\frac{1}{2}}} + \frac{1}{[(x-x')^2 + (y-y')^2 + (z+z')^2]^{\frac{1}{2}}} \\ & + \sum_{n\in Z\setminus\{0\}} \left(\frac{1}{[(x-x')^2 + (y-y')^2 + (z-(2nH+z'))^2]^{\frac{1}{2}}} \right. \\ & \left. + \frac{1}{[(x-x')^2 + (y-y')^2 + (z+(2nH+z'))^2]^{\frac{1}{2}}} - \frac{1}{|n|H} \right). \end{aligned}$$

Using the operator K, we can invert Eq. (4):

$$\psi = Kq. \tag{6}$$

3. Statement of the main result

Henceforth H denotes a fixed positive number and $\Omega = \mathbb{R}^2 \times (0, H)$. Points in $\mathbb{R}^3$ are denoted by $r = (x, y, z)$, $r' = (x', y', z')$. The open box $(-l, l) \times (-l, l) \times (\frac{1}{l}, H - \frac{1}{l})$ is denoted by Ω_l. We use the notation $B_R(r)$ for the open ball at point $r \in \mathbb{R}^n$ with radius $R > 0$. For $A \subset \mathbb{R}^n$; $n \in \mathbb{N}$, $|A|$ denotes the Lebesgue measure of A. We say $A \subset \mathbb{R}^n$, $n = 2, 3$ is dense